# Mastering NCERT Mathematics-Class IX

# Solved Exercises

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# Acknowledgements

Present book comes in existence to solve the dificulty of student class 9th of the subject mathematics.The book contains all the examples of NCERT mathematics book and their exercise. The solution of examples and exercise represented in simple and logical form that faciliate the students to understand the concepts and steps of solutions with proper framework. The study of this book will develop more skill to derive the answer of problems to reader accurately and efficiently.

This book covers whole syllabus of class $9^{th}$ of UP Board and other all board which fallow NCERT book of mathematics. It is also very helfull all competative exams. All the scholar who need to strong foundation of mathematics book properly guide him to study to next heigher classes. Some mind storming exercise also added in some chapter for crucial examination like Mathematics Olampiod for class $9^{th}$. Auther is not responsible to any harm or damage from book information, fact or by use, whhile every possible effort to make book errorless.

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# 1 NUMBER SYSTEM

## 1.1 Introduction

*Story of number:* When Yash wlaking at the bank of river, he collected the white stones. During the collection Yash tells, the number of stones to his sisther Lalita as; 1, 2, 3, 4, ..... This number which represent the stone of Yash that collected by him is called Natural number and it is denoted by $N$. Now when he transfer all stones to his sister then the stone left to his is represent by symble $'0'$ and called zero. The numbers which need to Yash to expresed stone is called whole number which denoted by $W$. Now Yash gave all stone to Lalita but Lalita is demanding one more stone because he was prommised to Laltita that he will give one more stone, when he will get. This last conversation that number represented by $-1$. So it is possible to find infinite number of numbers as natural number. The collection of all number which discussued above is called Integer and detoted as $I$ or $Z$.

The mathematicaly Natural number, Whole number, Integar denoted and written as:

- $N = \{1, 2, 3, 4, \dots\}$
- $W = \{0, , 2, 3, 4, 5, \dots\}$

- $Z = \{\cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$

**Properties of *Natural numbers*:**

i Sum of two ***natular numbers*** is always a ***natular number***.

ii Product of two ***natular numbers*** is always a ***natular number***.

iii Suntraction of two ***natular numbers*** need not be a ***natular numbers***.

iv division of two ***natular numbers*** need not be a ***natular number***.

**Properties of *whole number*:**

i Sum of two ***whole numbers*** is always a ***whole number***.

ii Product of two ***whole numbers*** is always a ***whole number***.

iii Subtraction of two ***whole numbers*** need not be a ***whole number***.

iv division of two ***whole numbers*** need not be a ***whole number***.

**Properties of *integer* :**

i Sum of two ***integers*** is always an ***integer***

ii Product of two ***integers*** is always an ***integer***

iii Suntraction of two ***integers*** always an ***integer***

iv division of two ***integers*** need not be an ***integer***

## 1.2 Rational numbers

We know that divison of any two integers need not be an integer, if we collect all integers and all numbers which is generated with division by two integers is called rational number and denoted by $Q$. Mathematicaly it is defined as,

**Definition 1.2.1.** A number $r$ is called a ***rational number***, if it can be written in the form $\frac{p}{q}$, where $p$ and $q$ are integer and $q \neq 0$.

**Example 1.2.2.** i $1, 2, 4, -1, -2, -3$

ii $\frac{3}{4}, \frac{8}{7}$

iii .2, .257, .24578950,

iv $\overline{.5}, \overline{.3578}$,

**Equivalent rational number:** The representation of a rational number is not unique, so the numbers that represent a rational number is called **equivalent rational numbers** or fractions.

**Example 1.2.3.** i $\frac{3}{5} = \frac{9}{15} = \frac{12}{20}$

ii $\frac{5}{4} = \frac{10}{8}$

**Remark:** Every rational number has infinite number of equivalent rational number.Then all equivalent rational numbers represent a rational number $\frac{p}{q}$, where $p$ and $q$ has co-prime or theire gcd is 1.

**Example 1.2.4.** Are the following example true or false ? Give reason for your answers.

i Every whole number is a natural number.

ii Every integer is a rational number.

iii Every rational number is an integer.

iv Every rational number is a natural number.

v Every natural number is an integer.

vi Every integer is a natural number.

**Solution:**

i False, because zero is whole number, but not a natural number.

ii True, because every integer $n$ can be expressed in the form $\frac{n}{1}$, which is a rational number.

iii False, at least one number $\frac{1}{2}$ is a rational but not integer.

iv False, because there is at least one number $\frac{1}{2}$ which is rational number, but not natural number.

v True,because every natural number is a positive integer.

vi False, because there is an integer $-1$, which is not a natural number.

**Method to find a rational number between two rational numbers:**

(1). Frist mathod to find a rartional number between two rational number.

**Example 1.2.5.** Find the five rational numbers between 1 and 2.

**Solution :**

We know that to find a rational number between $r$ and $s$, you can add $r$ and $s$ and divide the sum by 2, that is $\frac{r+s}{2}$ lise between $r$ and $s$. So, $\frac{3}{2}$ is a number between 1 and 2. Contune in this manner to find next four rational numbers between 1 and 2. These foue numbers $\frac{5}{4}, \frac{11}{8}, \frac{13}{8}, \frac{7}{4}$.

**(2)** Second method or One step method to find rational numbers betweet two rational number.

**Solution :** We want to five numbers between 1 and 2, we write 1 and 2 as equivalent rational with demoninator $5+1=6$, i.e., $1=\frac{6}{6}$ and $2=\frac{12}{6}$. Then write the numbers which numinatar are $6+1$, adding one by one upto 12 and denominator fixed 6. Get $\frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}, \frac{11}{5}$ five rational numbers between 1 and 2.

Let $a$ and $b$ two rational number, then $N$ rational number, say $r_n$ between $a$ and $b$ given as:

$$r_n = \frac{a\,(N+1) + n\,(b-a)}{N+1}$$

Where $n = 1, 2, 3, \ldots N$.

## 1.3 Solutin of Exercise 1.1

**Problem 1.3.1.** Is zero a rational number ? Can you write it in the form $\frac{p}{q}$, where $p$ and $q$ are integer and $q \neq 0$.

**Solution:** Yes, because 0 can be written as $\frac{0}{q}, q \neq 0$.

**Problem 1.3.2.** Find six rational numbers between 3 and 4.

**Solution:** We know that the formula to find $N$ rational number between $a$ and $b$ is

$$r_n = \frac{a\,(N+1) + n\,(b-a)}{N+1}$$

Where $n = 1, 2, 3, \ldots N$. So,

$$r_1 = \frac{3\,(6+1) + 1 \cdot 1}{6+1} = \frac{3 \cdot 7 + 1}{7} = \frac{22}{7}$$

Similarly

$$r_2 = \frac{23}{7}$$

$$r_3 = \frac{24}{7}$$

$$r_4 = \frac{25}{7}$$

$$r_5 = \frac{25}{7}$$

$$r_6 = \frac{26}{7}.$$

Therefore six rartional number between 3 and 4 are $\frac{22}{7}$, $\frac{23}{7}$, $\frac{24}{7}$, $\frac{25}{7}$, $\frac{26}{7}$, $\frac{27}{7}$.

**Problem 1.3.3.** Find five rational number between $\frac{3}{5}$ and $\frac{4}{5}$.

**Solution:** From second method, we have

$$r_n = \frac{a\,(N+1) + n\,(b-a)}{N+1}$$

Where $n = 1, 2, 3, \ldots N$.

$$r_1 = \frac{\frac{3}{5}\,(5+1) + 1\left(\frac{4}{5} - \frac{3}{5}\right)}{5+1} = \frac{\frac{3\times 6}{5} + 1\frac{1}{5}}{6} = \frac{19}{30}$$

Similarly

$$r_2 = \frac{20}{30}$$

$$r_3 = \frac{21}{30}$$

$$r_n = \frac{22}{30}$$

$$r_n = \frac{23}{30}$$

The desire rational numbers are $r_1 = \frac{19}{30},\ r_2 = \frac{2}{3},\ r_3 = \frac{7}{10},\ r_n = \frac{11}{15},\ r_n = \frac{23}{30}$.

**Problem 1.3.4.** State whether the following statements are true or false.

Give reasons for your answer.

i . Every whole is a natural number.

ii . Every integer is a whole number.

iii . Every rational number is a whole number,

**Solution:**

i . No, because 0 is not natural number but it belong to whole number.

ii . No, because negative integer are not included whole number.

iii . No, atleast one rational number as .2 is not whole number.

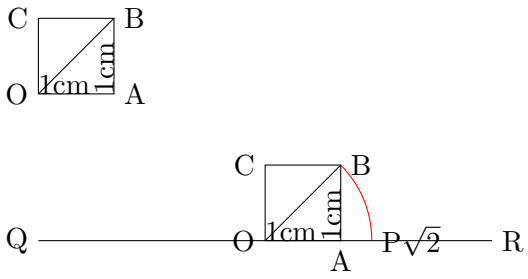
## 1.4 Irrational Numbers

**Definition 1.4.1.** A number 's' is called *irrational*, if it cannot be written in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.

**Some examples:** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{17}$, $\pi$, $0.10110111011110\ldots$.

Every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number.

**Example 1.4.2.** Locate $\sqrt{2}$ on the number line.

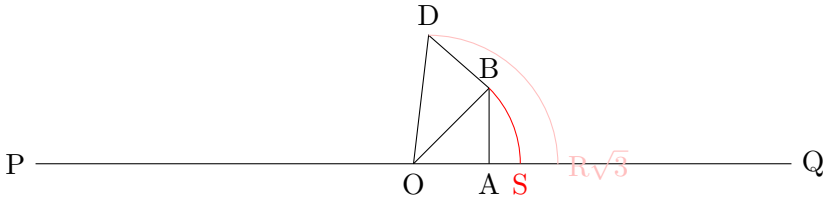



**Solution:** Consider a square OABC, with each side 1 unite length.

Consider a square OABC, with side 1 unit in length. Then according Pythagoras theorem the diagonal of square will be $\sqrt{2}$. Now identify the point whose distance equal to length OB, this point represent $\sqrt{2}$ on number line.

**Example 1.4.3.** Locate $\sqrt{3}$ on the number line.

**Solution:**




Construct AB of unit lenght perpendicular on line OA, which is also unit length. Therefore by Pythagoras theorem, we have $OB = \sqrt{1^2 + 1^2} = \sqrt{2}$. Similarally $OD = \sqrt{\sqrt{2}^2 + 1^2} = \sqrt{3}$. Now draw an arc with radius $\sqrt{3}$ cm at center 'O' which cut line segment PQ at R that represent $\sqrt{3}$ on number line.

## 1.5 Solution of Exercise 1.2

**Problem 1.5.1.** State whether the following statements are true or false. Justify your answers.

1. Every irrational number is a real number.
2. Every point on the number line is of the form $\sqrt{m}$, where $m$ is a natural number.
3. Every real number is a rational number.

**Solution:**

1. Yes, real number is collection of all rational and irrartional numbers.
2. No, $-1$ is not square root of any natural number.
3. No, real number contains rational and irrational numbers.

**Problem 1.5.2.** Are the square roots of all positive integers irrational ? If not, give an example of the square root of a number that is a rational number.

**Solution:** No, $\sqrt{4}$ is 2, which is a rational number.

**Problem 1.5.3.** Show how $\sqrt{5}$ can be represented on the number line.

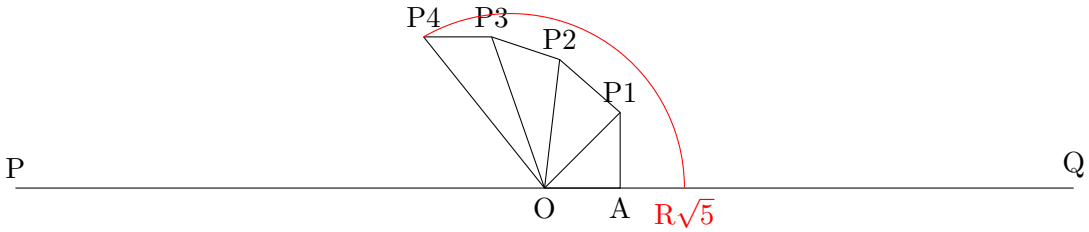



Figure 1.5.1

**Solution:**

Frist draw a right angle triangle $OAP_1$ with OA and $AP_1$ are unit length, then by Pythagoras theorem $OP_1 = \sqrt{2}$. Similar construction draw perpendicular line at hypotenus of each triangle of unit length, the line segment $OP_4 = \sqrt{5}$. Next draw an arc of radius $OP_4$ which meet at point R on PQ. This is require point that represent $\sqrt{5}$ on number line.

## 1.6 Real Number and Decimal Expantions

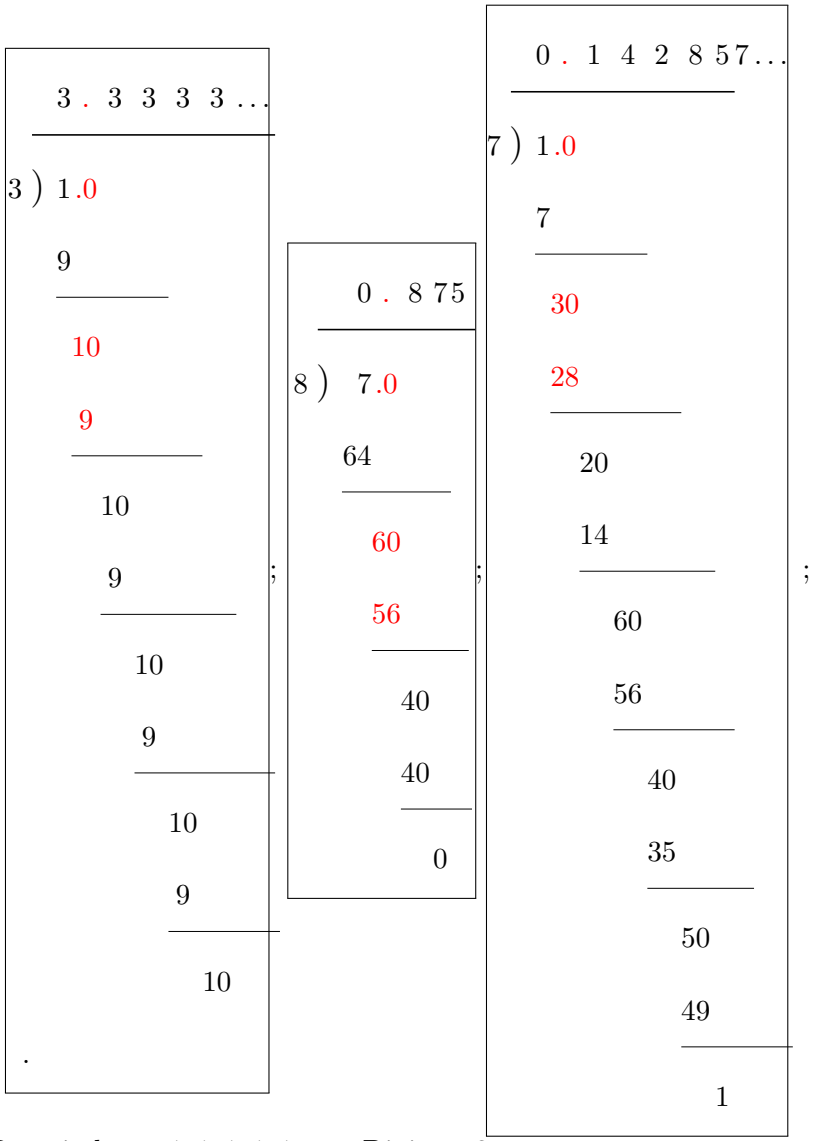
We are familiar with decimal number system, each number can be represent in decimal number, So first observe some rational numbers and their decimal represntation.

**Example 1.6.1.** Find the decimal expansions of

i $3\overline{)10}$

ii $7\overline{)8}$

iii $7\overline{)1}$ .




Remainders : $1, 1, 1, 1, 1, \ldots$, Divisor: 3,

Remainders : $6, 4, 0,$, Divisor: 8,

Remainders : $3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, \ldots$, Divisor: 7.

**Observation:** If you closely analysize the example (1.6.1), you get follow-

ing facts;

1. The remainder either becomes 0 after certain a stage, or start repeating themseves.

2. The number of entries in the repeating string of remainders is less than the divisior.

3. If the remainders repeat, then we get a repeating block of digits in the quotient.

The above fact true for all rational number which in the form $\frac{p}{q}, q \neq 0$. So on division of p by q two cases arise either remainder zero or never become zero.

**The remainder becomes zero:** When remainder becomes zero after a certain stage, then in the decimal representations of rational number the decimal number terminates or ends after a finite number of steps. This decimal representation is called `terminating decimal`.

**The remainder never becomes zero:** When remainder never becomes zero, then it repeats after a certain stage, so we find a repeating block of digits in quotion.This representation of decimal called `non terminating recurring decimal.`

**Example 1.6.2.** Show that 3.142678 is a rational number.

**Solution:** We have $3.142678 = \frac{3.142678}{1000000}$, hence it is a rational number.

**Example 1.6.3.** Show that $0.333\cdots = 0.\overline{3}$, can expressed in the form $\frac{p}{q}$ where $p$ and $q$ are integer and $q \neq 0$.

**Solution:** Let $x$ is $0.\overline{3}$. So,

$$x = 0.333\ldots$$

Now

$$10x = 10 \times (0.333\ldots) = 3.3333\ldots$$

next

$$10x = 3 + x$$

since $x = 0.333\ldots$,

Solving for $x$, we get

$$9x = 3, x = \frac{1}{3}$$

.

**Example 1.6.4.** Show that $1.27272727\cdots = 1.\overline{27}$ can be expressed in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.

**Solution:** Let $x = 1.\overline{27}$, here two digit are repeating, so we multiply $x$ by 100 to get,

$$100x = 1.272727\ldots$$

$$100x = 126 + 1.272727\cdots = 126 + x$$

So

$$100x - x = 126 \implies 99x = 126$$

$$x = \frac{126}{99} = \frac{14}{11}$$

.

**Example 1.6.5.** Show that $0.235353535\cdots = 0.2\overline{35}$ can be expressed in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$

**Solution:** Let $x = 0.2\overline{35}$, here 2 is not repeated but the block 35 repeating, so we multiply $x$ by 100 to get,

$$100x = 23.53535\ldots$$

$$100x = 23.3 + 0.23535\cdots = 23.3 + x$$

So

$$100x - x = 23.3 \implies 99x = 23.3$$

$$x = \frac{233}{990}$$

.

$$x = \frac{233}{990} = 0.2\overline{35}$$

.

**Result:** *The decimal expansion of rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expantion is terminating or non-terminating recurring is rational.*

From above discussion, we recognise the decimal expantion of rational number, but except rational number the decimal representation is `non-terminating non-recurring`.

**Result-2 :** *The decimal expantion of an irrational number is non-terminating non recurring. Moreover, a number whose decimal expantion is non-terminating non-recurring is irrational.*

**Example 1.6.6.** Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

**Solution :** We know that $\frac{1}{7} = 0.\overline{142857}$. So we can find observing the remainder of $\frac{1}{7} = 0$, we find $\frac{2}{7} = 0.\overline{285714}$. To find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$, means a number which is non-terminating non-recurring lying between them. One of them is $0.160160016000160000160000\ldots$.

## 1.7 Solution of Exercise 1.3

**Problem 1.7.1.** Write the following in decimal form and say what kind of decimal expantion each has :

i $\frac{36}{100}$,

ii $\frac{1}{11}$,

iii $4\frac{1}{8}$,

iv $\frac{3}{13}$,

v $\frac{2}{11}$,

vi $\frac{329}{400}$.

**Solution :**

i

$$
\begin{array}{r}
0.36 \\
\hline
100\,\big)\,360 \\
300 \\
\hline
600 \\
600 \\
\hline
0
\end{array}
$$

ii

$$
\begin{array}{r}
0.999\ldots \\
\hline
11\,\big)\,100 \\
99 \\
\hline
100 \\
99 \\
\hline
100 \\
99 \\
\hline
1
\end{array}
$$

The $\frac{36}{100} = .36$, represents terminating decimal next the fraction $\frac{1}{11} = .9999\ldots$, represents non terminating recurring decimal.

$$
\begin{array}{r}
4.125 \\
\hline
8\,\big)\,33 \\
32 \\
\hline
10 \\
8 \\
\hline
20 \\
16 \\
\hline
40 \\
40 \\
\hline
0
\end{array}
$$

iii

$$
\begin{array}{r}
0.2376153846\ldots \\
\hline
13\,\big)\,30 \\
26 \\
\hline
40 \\
39 \\
\hline
100 \\
91 \\
\hline
90 \\
78 \\
\hline
20 \\
13 \\
\hline
70 \\
65 \\
\hline
50 \\
39 \\
\hline
110 \\
104 \\
\hline
60 \\
52 \\
\hline
8\,0 \\
7\,8 \\
\hline
2
\end{array}
$$

iv

The fraction $4\frac{1}{8} = 4.125$ is representations of terminating decimal. The fraction $\frac{3}{13} = 0.2376153$ is representations of non-terminating recurring decimal.

v

$$
\begin{array}{r}
0.1818\ldots \\
\hline
11\,\big)\,20 \\
11 \\
\hline
90 \\
88 \\
\hline
20 \\
11 \\
\hline
90 \\
88 \\
\hline
2
\end{array}
$$

vi

$$
\begin{array}{r}
0.825 \\
\hline
400\,\big)\,3290 \\
3200 \\
\hline
900 \\
800 \\
\hline
2000 \\
2000 \\
\hline
0
\end{array}
$$

The fraction $\frac{2}{11} = 0.1818\ldots$ is representations of non-terminating recurring decimal. The fraction $\frac{429}{400} = 4.125$ is representations of terminating decimal.

**Problem 1.7.2.** You know that $\frac{1}{7} = 0.\overline{142857}$. Can you predict what the decimal expansions of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ are, without actually doing the long division ? If so, how ?

**Solution :**

After division of $\frac{1}{7} = 0.142857\ldots$ if you carefully studies the sequence of remainder and quotiont are corelated eachother as $1, 3, 2, 6, 4, 5$, and $1, 4, 2, 8, 5, 7$ repeated infinitaly times. So, the quotiont of fraction $\frac{2}{7}$ has contains in this ordered $.2, 8, 5, 7, 1, 4,$ because the remainders has been apears in this formate also. Hence $\frac{2}{7} = 0.857142\ldots$. Simillarly $\frac{3}{7} = 0.428571\ldots, \frac{4}{7} = 0.571428\ldots, \frac{5}{7} = 0.714285\ldots, \frac{6}{7} = 0.857142$ will be,

**Problem 1.7.3.** Express the following in the form $\frac{p}{q}$, where $p$ and $q$ are integer and $q \neq 0$.

i $0.\overline{6}$

ii $0.4\overline{7}$

iii $0.\overline{001}$

**Solution :**

i Let $x = 0.\overline{6}$. Since one digit is repeating, we multiply $x$ by 10 to get

$$10x = 6.666666\ldots$$

$$10x = 6 + 0.6666666\ldots$$

$$10x = 6 + x$$

$$9x = 6,$$

which gives

$$x = \frac{2}{3}$$

ii Let $x = 0.4\overline{7}$. Since one digit is repeating, we multiply $x$ by 10 to get

$$10x = 4.77777\ldots$$

$$10x = 4 + 0.3 + 0.4777777\ldots$$

$$10x = 4.3 + x$$

$$9x = 4.3,$$

which gives

$$x = \frac{43}{90}$$

iii Let $x = 0.\overline{001}$. Since three digit is repeating, we multiply $x$ by 1000 to get

$$1000x = 1.001001001\ldots$$

$$1000x = 1 + 0.001001001\ldots$$

$$1000x = 1 + x$$

$$999x = 1,$$

which gives

$$x = \frac{1}{999}$$

**Problem 1.7.4.** Express $0.99999\ldots$ in the form $\frac{p}{q}$, Are you surprised by your answer ? With your teacher and classmets discuss why ? the answer makes sense.

**Solution :**

Let $x = 0.\overline{9} = 0.999999\ldots$. Since one digit is repeating, we multiply $x$ by 10 to get

$$10x = 9.999999\ldots$$

$$10x = 9 + 0.999999\ldots$$

$$10x = 9 + x$$

$$9x = 9,$$

which gives

$$x = \frac{9}{9} = 1$$

.

**Problem 1.7.5.** What can be maximum number of digits be in the repeating block of digits in the decimal expansion of $\frac{1}{17}$ ? Perform the division to check your answer.

**Solution :** The maximunm number of digits in repeating block of decimal expantion of $\frac{1}{17}$ can be 16.

$$
\begin{array}{r}
0.0588235294117647 \\
17\overline{)\,100} \\
85 \\
\hline
150 \\
136 \\
\hline
140 \\
136 \\
\hline
40 \\
34 \\
\hline
60 \\
51 \\
\hline
90 \\
85 \\
\hline
50 \\
34 \\
\hline
160 \\
153 \\
\hline
70 \\
68 \\
\hline
20 \\
17 \\
\hline
30 \\
17 \\
\hline
130 \\
119 \\
\hline
110 \\
102 \\
\hline
80 \\
68 \\
\hline
120 \\
119 \\
\hline
1
\end{array}
$$

**Problem 1.7.6.** Write three numbers whose decimal expansions are non-terminating non-recurring.

**Solution :** The following decimal numbers are non-terminating non-recurring.

i $0.0100100010001000001\ldots$

ii $0.18018001800018000018\ldots$

iii $0.27027002700027000027\ldots$

**Problem 1.7.7.** Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.

**Solution :** We have two given numbers as $\frac{5}{7} = 0.\overline{71428}$ and $\frac{9}{11} = 0.\overline{81}$,then non-terminating and non-recurring decimal number between these two numbers give irrartional numbers that lise bwtwen them also. As

i $0.71071007100071000071\ldots$

ii $0.72072007200072000072\ldots$

iii $0.73073007300073000073\ldots$

**Problem 1.7.8.** Classify the following numbers as rational or irrational:

i $\sqrt{23}$

ii $\sqrt{225}$

iii 0.3796

iv $7.478478\ldots$

v $1.101001000100001\ldots$

**Solution:** In the given number following are rational numbers 0.3796, $7.478478\ldots$ and irrational numbers are $\sqrt{225}$, $\sqrt{23}$, $1.101001000100001\ldots$.

## 1.8 Operations and Real Numbers

You know that the rational numbers satisfy the `commutative, associative and distributive law for addition and multiplication.` Moreover, if we add, subtract, multiply or divide (except zero) two rational numbers, we still get a rational number. This property is called that rational numbers are **'closed'** with respect to addition, subtraction, multiplication and division.However the sum, difference and product of irrational numbers are need not irrartional numbers. We discuss many example to clearify this notion.

**Example 1.8.1.** Check whether $\left(\sqrt{6}\right)+\left(-\sqrt{6}\right), \left(\sqrt{3}\right).\left(\sqrt{3}\right)$, and $\dfrac{\sqrt{17}}{\sqrt{17}}$ are rational or irrational numbers.

**Solution:**

$$\left(\sqrt{6}\right) + \left(-\sqrt{6}\right) = \sqrt{6} - \sqrt{6} = 0,$$

$$\left(\sqrt{3}\right).\left(\sqrt{3}\right) = \sqrt{3}.\sqrt{3} = 3,$$

$$\frac{\sqrt{17}}{\sqrt{17}} = 1.$$

All these are integer. So, all these are rational numbers.

**Example 1.8.2.** Check whether $7\sqrt{5}, \frac{7}{\sqrt{5}}, \sqrt{2}+21, \pi-2$ are irrational or not.

**Solution:** We know that $\sqrt{5} = 2.236\ldots, \sqrt{2} = 1.1412\ldots, \pi = 3.1415\ldots$

Then $7\sqrt{5} = 15.652\ldots$, $\frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{7\sqrt{5}}{5} = 3.1304\ldots$

$\sqrt{2} + 21 = 22.4142\ldots$, $\pi - 2 = 1.1415\ldots$ All these are non-terminating non-recurring decimals. So, all these are irrarional numbers.

**Result:**

i The sum, difference and product of irrational numbers are need not be an irrartional number.

ii The *nth* root of an positive irrartional number need not be an irrational number.

**Example 1.8.3.** Add $2\sqrt{2} + 5\sqrt{3}$ and $\sqrt{2} - 3\sqrt{3}$.

**Solution:**

$$\left(2\sqrt{2} + 5\sqrt{5}\right) + \left(\sqrt{2} - 3\sqrt{3}\right) = \left(2\sqrt{2} + \sqrt{2}\right) + \left(5\sqrt{3} - 3\sqrt{3}\right)$$

$$= (2+1)\sqrt{2} + (5-3)\sqrt{3} = 3\sqrt{2} + 2\sqrt{3}$$

**Example 1.8.4.** Multiply $6\sqrt{5}$ by $2\sqrt{5}$.

**Solution:** $6\sqrt{5} \times 2\sqrt{5} = 6 \times 2 \times \sqrt{5} \times \sqrt{5} = 12 \times 5 = 60$

**Example 1.8.5.** Divide $8\sqrt{15}$ by $2\sqrt{3}$.

**Solution:** $\dfrac{8\sqrt{15}}{2\sqrt{3}} = \dfrac{8\sqrt{3}\sqrt{5}}{2\sqrt{3}} = 4\sqrt{5}$

From these example we can conclude the followinf facts

i The sum or difference of a rational number and irrational numbers is an irrational number.

ii The product or quotient of a non zero rational number with an irrational number is an irrational number.

iii If we add, subtract, multiply or divide two irrational numbers the result may be rational or irrational number.

### Squra root of positive rational number

Let $a > 0$ be a real number. Then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$. Now we will show, how to find $\sqrt{x}$ for any given positive real number $x$ geometricaly. More generaly, to find $\sqrt{x}$ for any positive real number $x$, we mark B so that

$AB = x$ units, and as in Fig.(1.8.1) mark C so that $BC = 1$ unit.Then we find $BD = \sqrt{x}$. We can prove this result using the pythagoras Theorem.

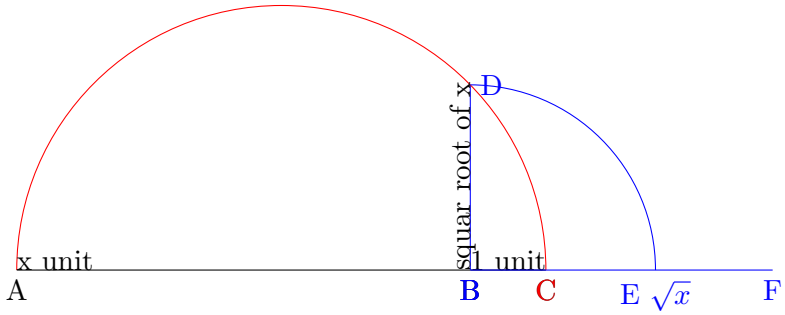



Figure 1.8.1: Representation of $\sqrt{x}$

### Identites related to squar root

Let $a$ and $b$ be positive real numbers. Then

i $\sqrt{ab} = \sqrt{a}\sqrt{b}$

ii $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

iii $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$

iv $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$

v $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{c} + \sqrt{d}\right) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$

vi $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

**Example 1.8.6.** Simplfy the following expressions:

i $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)$

ii $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

iii $\left(\sqrt{3}+\sqrt{7}\right)^2$

iv $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}-\sqrt{7}\right)$

**Solution:**

i $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right) = 10+5\sqrt{5}+2\sqrt{7}+\sqrt{35}$

ii $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right) = 5^2-\left(\sqrt{5}\right)^2 = 25-5 = 20$

iii $\left(\sqrt{3}+\sqrt{7}\right)^2 = \left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{7}+\left(\sqrt{7}\right)^2 = 3+2\sqrt{21}+7 = 10+2\sqrt{21}$

iv $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}-\sqrt{7}\right) = \left(\sqrt{11}\right)^2-\left(\sqrt{7}\right)^2 = 11-7 = 4$

**Remark:** Note that after 'simplification'the above example, we get that the expression should be written as the sum of a rational and an irrational number. Lte us see, if we can 'rationalise the denomenator of a fraction,that maens to make the denominator into a rational number.

**Example 1.8.7.** Rationalise the denominator of $\frac{1}{\sqrt{2}}$.

**Solution:** We want to write $\frac{1}{\sqrt{2}}$ as an equivalent fraction in which the denominator is rational number. We know that $\sqrt{2}.\sqrt{2}$ is a rational number. Also we know that multiplication non zero number in numenator and

denominator gives an equivalent fraction. So, cosidaring these fact we get,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Now it is easy to locate $\frac{1}{\sqrt{2}}$ on number line.

**Example 1.8.8.** Rationalise the denominator of $\frac{1}{2-\sqrt{3}}$.

**Solution:** To rationalise the $\frac{1}{2-\sqrt{3}}$, multiply and divide by $2-\sqrt{3}$ to get $\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$.

**Example 1.8.9.** Rationalise the denominator of $\frac{5}{\sqrt{3}-\sqrt{5}}$.

**Solution:** Here

$$\frac{5}{\sqrt{3}-\sqrt{5}} = \frac{5}{\sqrt{3}-\sqrt{5}} \times \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}} = \frac{5\left(\sqrt{3}+\sqrt{5}\right)}{3-5} = \frac{-5}{2}\left(\sqrt{3}+\sqrt{5}\right).$$

**Example 1.8.10.** Rationalise the denominator of $\frac{1}{7+3\sqrt{2}}$.

**Solution:**

$$\frac{1}{7+3\sqrt{2}} = \frac{1}{7+3\sqrt{2}} \times \frac{7-3\sqrt{2}}{7-3\sqrt{2}} = \frac{7-3\sqrt{2}}{49-18} = \frac{7-3\sqrt{2}}{31}.$$

The process of converting a real number to an equivalent expression whose denominator is a rational number is called `rationalising the denominator`.

## 1.9 Solution of Exercise 1.4

**Problem 1.9.1.** Classsify the following numbers as rational or irrational:

i $2-\sqrt{5}$

ii $\left(3+\sqrt{23}\right)-\sqrt{23}$

iii $\dfrac{2\sqrt{7}}{7\sqrt{7}}$

iv $\dfrac{1}{\sqrt{2}}$

v $2\pi$

**Solution:**

i "Irrational number". The decimal representation of $2-\sqrt{5}$ is non terminating non-recurring decimal.

ii "Rational number". After simplify of $\left(3+\sqrt{23}\right)-\sqrt{23}=3$, which is a rational number.

iii "Rational number". After simplify $\dfrac{2\sqrt{7}}{7\sqrt{7}}=\dfrac{2}{7}$, which is a rational number.

iv "Irrational number". After simplify $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$, which gives non terminating non-recurring decimal represention.

v "Irrational number". It also represents non terminating non-recurring decimal.

**Problem 1.9.2.** Simplify each of the following expression:

i

$$\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$$

ii

$$\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$$

iii

$$\left(\sqrt{5}+\sqrt{2}\right)^2$$

iv

$$\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$$

**Solution:**

i $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right) = 3\times 2+3\sqrt{2}+2\sqrt{3}+\sqrt{2}\sqrt{3} = 6+3\sqrt{2}+2\sqrt{3}+\sqrt{2}\sqrt{3}$

According identities (v) we simplify the problems.

ii Applying identies iv, $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right) = 3^2-\left(\sqrt{3}\right)^2 = 9-3 = 6$

iii Applying identies vi, $\left(\sqrt{5}+\sqrt{2}\right)^2 = \left(\sqrt{5}\right)^2+2\sqrt{5}\sqrt{2}+\left(\sqrt{2}\right)^2 = 5+\sqrt{10}+2 = 7+\sqrt{10}$

iv Applying identies iii, $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)=\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2=5-2=3$

**Problem 1.9.3.** Recall, $\pi$ is defined as the ratio of the circumference (say c) of a circle to its diameter (say d). That is $\pi=\frac{c}{d}$. This seems to contradict the fact that $\pi$ is irrational. How will you resolve this contradiction ?

**Solution:** The ratio of circumference (say c) and diameter (say d) equivalt to $\pi$ as $\pi\sim\frac{c}{d}$.

**Problem 1.9.4.** Represent $\sqrt{9.3}$ on number line.

**Solution:** Draw the line $AB=9.3$ cm add $BC=1$ cm, then draw semicircle as diameter $AC=10.3$ and perpendicular line $BD$ where point $D$ on semicircle. We get required $BD=\sqrt{9.3}$ cm.

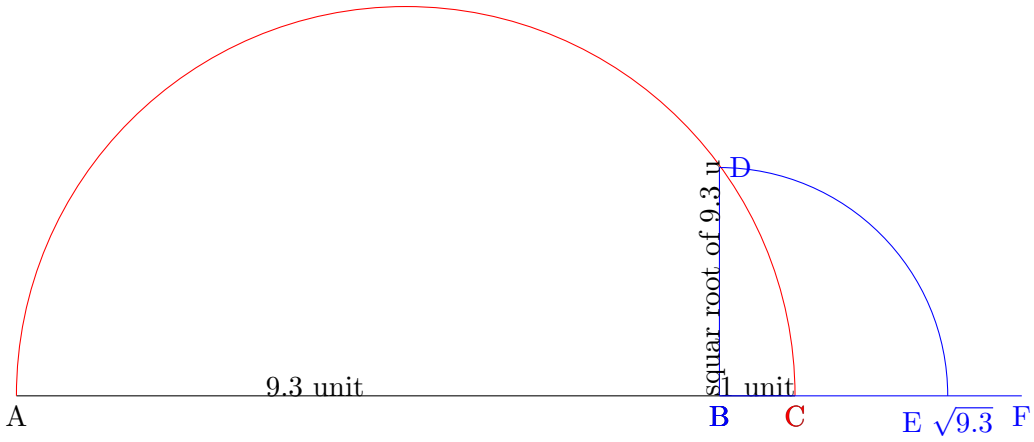



Figure 1.9.1: Reprentation of $\sqrt{9.3}$ on number line

**Problem 1.9.5.** Rationalise the denominators of the following:

i $\dfrac{1}{\sqrt{7}}$

ii $\dfrac{1}{\sqrt{7}-\sqrt{6}}$

iii $\dfrac{1}{\sqrt{5}+\sqrt{2}}$

iv $\dfrac{1}{\sqrt{7}-2}$

**Solution:**

i $\dfrac{1}{\sqrt{7}} = \dfrac{1}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{7}$

ii $\dfrac{1}{\sqrt{7}-\sqrt{6}} = \dfrac{1}{\sqrt{7}-\sqrt{6}} \times \dfrac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}} = \dfrac{\sqrt{7}+\sqrt{6}}{7-6} = \sqrt{7}+\sqrt{6}$

iii $\dfrac{1}{\sqrt{5}+\sqrt{2}} = \dfrac{1}{\sqrt{5}+\sqrt{2}} \times \dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \dfrac{\sqrt{5}-\sqrt{2}}{5-2} = \dfrac{\sqrt{5}-\sqrt{2}}{3}$

iv $\dfrac{1}{\sqrt{7}-2} = \dfrac{1}{\sqrt{7}-2} \times \dfrac{\sqrt{7}+2}{\sqrt{7}+2} = \dfrac{\sqrt{7}+2}{7-4} = \dfrac{\sqrt{7}+2}{3}$

**Remark:** The conjugate of $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$. To rationalise denominator, multiply conjugate of denominator in numenator and denominator then simplify it.

## 1.10 Law of Exponent for Real Numbers

In expression $a^n$, $a$ is a rational number and $n$ is a whole numbers. For the espression $a^n$, $a$ called the base and $n$ are the exponents. The exponent follow the following rules

i $a^m.a^n = a^{m+n}$

ii $(a^m)^n = a^{mn}$

iii $\dfrac{a^m}{a^n} = a^{m-n}, m > n$

iv $a^m b^m = (ab)^m$

v $\dfrac{1}{a^n} = a^{-n}$

vi $a^0 = 1$

**Example 1.10.1.** See the following

i $17^2.17^{-5} = 17^{2-5} = 17^{-3} = \dfrac{1}{17^3}$

ii $\left(5^2\right)^{-7} = a^{2\times(-7)} = 5^{-14}$

iii $\dfrac{23^{-10}}{23^7} = 23^{-10-7} = 23^{-17},$

iv $(7)^{-3}.(9)^{-3} = (7.9)^{-3} = (63)^{-3}$

## 1.11 Law of Extended Exponents for Real Numbers

Now we define $\sqrt[n]{a}$ for a real number $a > 0$ as follow: Let $a > 0$ be a real number and $n$ a positive integer. Then $\sqrt[a]{n} = b$, if $b^n = a$ and $b > 0$. In the language of exponents, we define $\sqrt[n]{a} = a^{\frac{1}{n}}$. So, in particuler $\sqrt[3]{5} = 3^{\frac{1}{5}}$.

Let $a > 0$ be a real number. Let $m$ and $n$ be integer such that $m$ and $n$ have no common factor other than 1 and $n > 0$. Then

$$a^{\frac{m}{n}} = \left(\sqrt[a]{n}\right)^n = \sqrt[n]{a^m}$$

Now we have following extended law of exponents:

Let $a > 0$ be a real number and $p$ and $q$ be rational numbers. Then, we have

i $a^p.a^q = a^{p+q}$

ii $(a^p)^q = a^{pq}$

iii $\dfrac{a^p}{a^q} = a^{p-q}$

iv $a^p b^p = (ab)^p$

**Example 1.11.1.** Simplify

i $2^{\frac{2}{3}}.2^{\frac{2}{3}}$

ii $\left(3^{\frac{1}{5}}\right)^4$

iii $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$

iv $13^{\frac{1}{5}}.17^{\frac{1}{5}}$

**Solution:** Simplify

i By law i,

$$2^{\frac{2}{3}}.2^{\frac{2}{3}} = 2^{\frac{2}{3}+\frac{1}{3}} = 2^{\frac{3}{3}} = 2^1 = 2$$

ii By law ii

$$\left(3^{\frac{1}{5}}\right)^4 = 3^{\frac{4}{5}}$$

iii By law iii

$$\frac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}} = 7^{\frac{1}{5}-\frac{1}{3}} = 7^{\frac{3-5}{15}} = 7^{\frac{-2}{15}}$$

iv By law of iv,

$$13^{\frac{1}{5}}.17^{\frac{1}{5}} = (13\times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$$

## 1.12 Solution of Exercise 1.5

**Problem 1.12.1.** i $64^{\frac{1}{2}}$

ii $32^{\frac{1}{5}}$

iii $125^{\frac{1}{3}}$

**Solution:**

i $64^{\frac{1}{2}} = \left(8^2\right)^{\frac{1}{2}} = 8^{2\times\frac{1}{2}} = 8^1 = 8$

ii $32^{\frac{1}{5}} = \left(2^5\right)^{\frac{1}{5}} = 2^{5\times\frac{1}{5}} = 2^1 = 2$

iii $125^{\frac{1}{3}} = \left(5^3\right)^{\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5^1 = 5$

**Problem 1.12.2.** Find:

i $9^{\frac{3}{2}}$

ii $32^{\frac{2}{5}}$

iii $16^{\frac{3}{4}}$

iv $125^{-\frac{1}{3}}$

**Solution:**

i $9^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}} = 3^{2\times\frac{3}{2}} = 3^3 = 27$

ii $32^{\frac{2}{5}} = \left(2^5\right)^{\frac{2}{5}} = 2^{5\times\frac{2}{5}} = 2^2 = 4$

iii $16^{\frac{3}{4}} = \left(2^4\right)^{\frac{3}{4}} = 2^{4\times\frac{3}{4}} = 2^3 = 8$

iv $125^{-\frac{1}{3}} = \left(5^3\right)^{-\frac{1}{3}} = 5^{3\times\left(-\frac{1}{3}\right)} = 5^{-1} = \frac{1}{5}$

**Problem 1.12.3.** Simlify:

i $2^{\frac{2}{3}}.2^{\frac{1}{5}}$

ii $\left(\frac{1}{3^3}\right)^7$

iii $\frac{11^{\frac{1}{2}}}{11^{\frac{1}{4}}}$

iv $7^{\frac{1}{2}}.8^{\frac{1}{2}}$

**Solution:**

i $2^{\frac{2}{3}}.2^{\frac{1}{5}} = 2^{\frac{2}{3}+\frac{1}{5}} = 2^{\frac{10+3}{15}} = 2^{\frac{13}{15}}$

ii $\left(\frac{1}{3^3}\right)^7 = \left(3^{-3}\right)^7 = 3^{-3\times 7 = 3^{-21}}$

iii $\frac{11^{\frac{1}{2}}}{11^{\frac{1}{4}}} = 11^{\frac{1}{2}-\frac{1}{4}} = 11^{\frac{2-1}{4}} = 11^{-\frac{1}{4}}$

iv $7^{\frac{1}{2}}.8^{\frac{1}{2}} = (7\times 8)^{\frac{1}{2}} = 56^{\frac{1}{2}}$

# 2 POLYNOMIALS

## 2.1 Introduction

In this chapter, we shall study with a particular type of algebraic expression, called *polynamial,* and the *terminology* related to it. We shall also study the *Remainder Theorem* and *Factor Theorem* and their use in the factorisation of polynomials. Next we shall study the algebraic identites and their use in factorisation and evaluating some given expressions.

## 2.2 Polynomials in One Variable

All the expression are of the form (a constant) $\times$ x . Now suppose we want to write an expression which is (a constant) $\times$ (a variable) and we do not know what the constant is. In such cases, we write the constant as $a, b, c$ etc. So the expression will be $ax$, say. The values of the constants remain the same throughout a particular situation, that is the values of the constants do not change in a given problem, but the value of a variable can keep changing.

Now consider your roof shape is a squre and length of side is 5 unit. What is its perimeter ? Exactly it is $4 \times 5 = 20$ unit. Similarly the area of your roof is $5^2 = 25$ squar unit. If length side of your roof is variable $x$, then perimeter and area will be $4x$ unit and $x^2$ unit. Now we can write other many example

like this as $2x + 5$, $x^2 + 3x + 1$, $x^3 + x^2 + 9x + 1$, these algebraic expression has an intresting thing that the power of variable is non negative integer. The algebraic expression in which the power of variable is a non negative integer (whole number) is called *polynamials*. The polynomial which contains only one variable is called *polynomials of one variable*. The expression in a polynamial which is seperated by $+$ or $-$ are called a **terms** of polynomial.

In the polynomial $x^3 + x^2 + 9x + 1$, the $x^3$, $x^2$, and $9x$, 1 are the terms of polynamial. The each terms of a polynomial can be express as multiplication of a real number and variable with power (whole number), this real number is called **ceofficients** of variable with power (whole number). Here the coefficient of $x^3$, $x^2$, and $x$, $x^0$ are $1, 1, 9, 1$ in the polynomial $x^3 + x^2 + 9x + 1$. Now the 2 is also a polynomial and 2 is the ceofficient of $x^0$.

**Example 2.2.1.** Categaries the following algebraic expression in polynamial and non polynomial.

- $2 + x$, this is polynomial.
- $\frac{1}{x}$, this is not polynomial because the power of variable negative integer $-1$.
- $\sqrt[3]{x} + x^2$ is not a polynomial because the power of variable $\frac{1}{3}$ which is not non negative integer.

## 2.3 Type of polynomial

**Type of polynomial on the basis of term:**

1. A polynamial in which no variable seems to apear is called **constant polynamial**, as 1, $-1$, 0 etc.

2. A constant polynamial which is always 0 is called **zero polynomial**, as 0.

3. The polynomials having only one term are called **monomials**, as $x$, $2x$.

4. A polynomials having only two terms is called **binomial**, as $x + 1$.

5. apolynomials having only three terms is called **trinomial**, as $x^2+3x+5$.

**Remark:**

1. A polynamial have only finite terms.

2. Each zero polynomial is a constant polynamail but every constant polynomialis not zero polynamial.

3. Each constant polynomial is monomial.

**Degree of polynomial:** The highest power of the variable present in a polynamial is called the *degree of the polynamial*.

**Example 2.3.1.** Find the degree of each polynomial

i $x^5 + x^4 + 3$

ii $2 - y^2 - y^3 + 2y^8$

iii 3

**Solution:**

i The highest power of the variable is 5. so the degree of the polynomial is 5.

ii The highest power of the variable is 8. so the degree of the polynomial is 8.

iii The term in the polynomial is only 3 which can be written as $3x^0$. So the exponent of $x$ is 0. therefore, the degree of polynamial is 0.

## Type of polynomial on the basis of degree:

1. A polynamial of dgree one is called **linear polynomial** as $x + 1$, $x$. The general form of a linear polynamials in one variable is $ax + b$, where $a \neq 0, b$ are real numbers.

2. A polynamial of degree two is called a **quadratic polynomial**, as $2x^2 + x + \pi$, $x^2 + 8$, $x^2$. In one variable the gerernal form of aqudratic polynomial is $ax^2 + bx + c$, where $a \neq 0$ and $b, c$ are real numbers.

3. A polynomial of degree three is called **cubic polynomial**, as $\sqrt[3]{2}x^3 + x^2 + x + 2$, $x^3 + \pi x^2 + 3x + 9$, $x^3$, $x^3 + x^2$. In one variable the general form of cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and $b, c, d$ are real numbers.

**Remark:**

i The degree of zero polynomial is not define.

ii The linear polynomial having at most two terms.

iii The qudratic polynomial having at most three terms.

iv The cubic polynomial having at most four terms.

## 2.4 Solution of Exercise 2.1

**Problem 2.4.1.** Which of the following expressions are polynomials in one variable and which are not ? State reason for your answer.

i $4x^2 - 3x + 7$

ii $y^2 + \sqrt{2}$

iii $3\sqrt{t} + t\sqrt{2}$

iv $y + \frac{1}{y}$

v $x^1 0 + y^3 + t^5 0$

**Solution**

i It is a polynomial in one variable. The exponent of variable is non negative integer.

ii It is a polynomial in one variable. The exponent of variable is non negative integer.

iii It is not a polynomial in one variable. The exponent of variable is fraction.

iv It is not a polynomial in one variable. The exponent of variable is negative integer $(-1)$.

v It is not a polynomial in one variable. There are three variable present in polynomial so, it is polynomial in three variable.

**Problem 2.4.2.** Write the coefficient of $x^2$ in each of the following:

i $2 + x^2 + x$

ii $2 - x^2 + x^3$

iii $\frac{\pi}{2}x^2 + x$

iv $\sqrt{2}x - 1$

**Solution:**

i The coefficient of $x^2$ is 1.

ii The coefficient of $x^2$ is $-1$.

iii The coefficient of $x^2$ is $\frac{\pi}{2}$.

iv The coefficient of $x^2$ is 0.

**Problem 2.4.3.** Give one example each of a binomial of degree 35 and of a momomial of degree 100.

**Solution:** The general form of binomials of degree 35 is $ax^{35} + b$, where $a, b \neq 0$ are real numbers. So polynomial is like $x^{35} + 1$. The general form of monomials of degree 100 is $ax^{100}$, where $a \neq 0$ is real numbers. So polynomial is like $x^{100}$.

**Problem 2.4.4.** Write the degree of each of the following polynomials:

| | *Polynomial* | *Degree* |
|---|---|---|
| $(i)$ | $5x^3 + 4x^2 + 7x$ | 3 |
| $(ii)$ | $4 - y^2$ | 2 |
| $(iii)$ | $5t - \sqrt{7}$ | 1 |
| $(iv)$ | 3 | 0 |

**Problem 2.4.5.** Classify the following as linear, quadratic and cubic polynomials:

i $x^2 + x$

ii $x - x^3$

iii $y + y^2 + 4$

iv $1 + x$

v $3t$

vi $r^2$

vii $7x^3$

**Solution:**

i Linear,the polynomial $(iv)$ and $(v)$ are linear polynomials.

ii Quadratic, the polynomial $(i), (iii), (vi)$ are quadratic polynomials.

iii Cubic, the polynomial $(ii), (vii)$ are cubic polynomials.

## 2.5 Zero of a Polynomial

Consider the polynomial $p(x) = 2x^2 - 3x + 1$. If we put $x = 2$ in $p(x)$, we get

$$p(1) = 2 \times 2^2 - 3 \times 2 + 1$$

$$= 2 \times 4 - 6 + 1$$

$$= 8 - 5$$

$$= 3$$

This number 3 is called value of polynomial $p(x)$ at $x = 2$.

### Value of a polynomial

Let $p(x)$ is polynomial in one variable $x$, then $p(k)$ is called *value* of polynomial at $x = k$, where $k$ is any rael number. Now for which $x = k$ the value of polynomial $p(k)$ is zero, then $x = k$ is called **zero of polynomial**.

**Remark:**

i Zero of polynomial may not exist for each polynomial.

ii Zero of polynomial is not unique.

**Example 2.5.1.** Check whether $-2$ and $-1, 2$ are zeroes of the polynomial $x + 2$.

**Solution:** Let $p(x) = x+2$. Then $p(-2) = -2+2 = 0, p(-1) = -1+2 = 1$, $p(2) = 2 + 2 = 4$, Therefore, $-2$ is a zero of the polynomial $x + 2$, but $-1, 2$ are not.

**Example 2.5.2.** Find the zero of polynomial $p(x) = 3x + 1$.

**Solution:** Now we equate the polynomial $p(x)$ is zero and find out the value $x$. $p(x) = 0$, $3x + 1 = 0$, $3x = -1$, $x = \frac{1}{3}$. So, the zero of polynomial is $-\frac{1}{3}$.

**Example 2.5.3.** Verify whether 3 and 0 are zeroes of the polynomial $x^2 - 3x$.

**Solution:** We have polynomial $p(x) = x^2 - 3x$.

Then

$$p(3) = 3^2 - 3 = 9 - 9 = 0$$

$$p(0) = 0^2 - 3.0 = 0 - 0 = 0$$

Hence 3 and 0 bothe are zeroes of the polynomial $x^2 - 3x$.

## 2.6 Solution of Exercise 2.2

**Problem 2.6.1.** Find the value of the polynomial $5x - 4x^2 + 3$ at

i $x = 0$,

ii $x = -1$,

iii $x = 2$

**Solution:** Let $p(x) = 5x - 4x^2 + 3$

i $p(0) = 5.0 - 40^2 + 3 = 0 + 0 + 3 = 3$

ii $p(-1) = 5 \times (-1) - 4(-1)^2 + 3 = -5 - 4 + 3 = -9 + 3 = -6$

iii $p(2) = 5.2 - 4.2^2 + 3 = 10 - 4.4 + 3 = 10 - 16 + 3 = -3$

**Problem 2.6.2.** Find the value $p(0), p(1)$ and $p(2)$ for each of the following polynomial:

i $p(y) = y^2 - y = 1$

ii $p(t) = 2 + t + 2t^2 - t^3$

iii $p(x) = x^3$

iv $p(x) = (x - 1)(x + 1)$

**Solution:**

i Let $p(y) = y^2 - y + 1$,

$$p(0) = 0^2 - 0 + 1 = 1$$

$$p(1) = 1^2 - 1 + 1 = 1 + 0 = 1$$

$$p(2) = 2^2 - 2 + 1 = 4 - 2 + 1 = 3$$

ii Let $p(t) = 2 + t + 2t^2 - t^3$

$$p(0) = 2 + 0 + 2.0^2 - 0^3 = 2 + 0 + 0 - 0 = 2$$

$$p(1) = 2 + 1 + 2.1^2 - 1^3 = 2 + 1 + 2 - 1 = 4$$

$$p(2) = 2 + 2 + 2.2^2 - 2^3 = 4 + 2.4 - 8 = 4 + 8 - 8 = 4$$

iii Let $p(x) = x^3$

$$p(0) = 0^3 = 0$$

$$p(1) = 1^3 = 1$$

$$p(2) = 2^3 = 8$$

iv Let $p(x) = (x - 1)(x + 1)$

$$p(0) = (0 - 1)(0 + 1) = (-1).1 = -1$$

$$p(1) = (1 - 1)(1 + 1) = 0.1 = 0$$

$$p(2) = (2 - 1)(2 + 1) = 1.3 = 3$$

**Problem 2.6.3.** Verify whether the following are zeroes of the polynomial, indicated againist them.

i $p(x) = 4x + 1, x = -\frac{1}{3}$. Next

$$p\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right) + 1 = -1 + 1 = 0$$

hence $-\frac{1}{3}$ is zero of polynomial.

ii Let $p(x) = 5x - \pi, x = \frac{4}{5}$, next

$$p\left(\frac{4}{5}\right) = 5.\frac{4}{5} - \pi = 4 - \pi$$

hence $\frac{4}{5}$ is not zero of the polynomial.

iii Let $p(x) = x^2 - 1, x = -1, 1$.

Now,

$$p(1) = 1^2 - 1 = 1 - 1 = 0$$

$$p(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

Hence $1, -1$ both are zeroes of polynomial.

iv Let $p(x) = (x + 1)(x - 2), x = -1, 2$ Next

$$p(-1) = (-1 + 1)(-1 - 2) = 0.(-3) = 0$$

$$p(2) = (2 + 1)(2 - 2) = 3.0 = 0$$

Hence $-1, 2$ bothe are zeroes of the polynomial.

v Let $p(x) = x^2, x = 0$ Next

$$p(0) = 0^2 = 0$$

Hence 0 is the zero of polynomial.

vi Let $p(x) = lx + m, x = -\frac{m}{l}$, Next

$$p\left(-\frac{m}{l}\right) = l\left(-\frac{m}{l}\right) + m = -m + m = 0$$

Hence $-\frac{m}{l}$ is the zero of the polynomial.

vii Let $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ Next

$$p\left(-\frac{1}{\sqrt{3}}\right) = 3\left(-\frac{1}{\sqrt{3}}\right)^2 - 1 = 3.\frac{1}{3} - 1 = 1 - 1 = 0$$

$$p\left(\frac{2}{\sqrt{3}}\right) = 3\left(\frac{2}{\sqrt{3}}\right)^2 - 1 = 3\frac{4}{3} - 1 \neq 0$$

Hence $-\frac{1}{\sqrt{3}}$ is the zeroes of polynomial but $\frac{2}{\sqrt{3}}$ is not zero of polynomial.

viii Let $p(x) = 2x + 1, x = \frac{1}{2}$. Next

$$p\left(\frac{1}{2}\right) = 2\frac{1}{2} + = 1 + 1 = 2$$

Hence $\frac{1}{2}$ is not zero of polynomial.

**Problem 2.6.4.** Find the zero of the polynomial in each of the following cases:

i Let $p(x) = x + 5$, consider $k$ is the zero of the polynomial $p(x)$, then $p(k) = 0$.

$$p(k) = 0$$

$$k + 5 = 0 \implies k = -5$$

Hence zero of the $p(x)$ is $-5$.

ii Simirally $(i)$, let $p(x) = x - 5$

$$p(k) = 0 \implies k - 5 = 0 \implies k = 5$$

So zero of $p(x)$ is 5.

iii Simirally $(i)$, let $p(x) = 2x + 5$

$$p(k) = 0 \implies 2k + 5 = 0 \implies 2k = -5 \implies k = \frac{-5}{2}$$

. So zero of $p(x)$ is $-\frac{5}{2}$

iv Similarally $(i)$ let $p(x) = 3x - 2$,

$$p(k) = 0 \implies 3k - 2 = 0 \implies 3k = 2 \implies k = \frac{2}{3}$$

So the zero of $p(x)$ is $\frac{2}{3}$.

v Similarally $(i)$, let $p(x) = 3x$,

$$p(k) = 0 \implies 3k = 0 \implies k = 0$$

So 0 is the zero of $p(x)$.

vi Similarally $(i)$, let $p(x) = ax, a \neq 0$,

$$p(k) = 0 \implies ak = 0 \implies k = 0$$

So 0 is zero of $p(x)$.

vii Similarally $(i)$, let $p(x) = cx + d, c \neq 0, c, d$ are real numbers.

$$p(k) = 0 \implies ck + d = 0 \implies ck = -d \implies k = -\frac{d}{c}$$

So $-\frac{d}{c}$ is zero of $p(x)$.

## 2.7 Factorisation of Polynomials

In this section we discuss that when we have a polynomial $p(x)$ more than one degree then how can we write the polynomial $p(x)$ as product of two or more polynomials. This process is called factorisation of polynomials.

**Theorem 2.7.1Factor Theorem:** *If $p(x)$ is a polynamial of degree $n \geq 1$ and $a$ is any real number, then $(i)$ $x - a$ is a factor of $p(x)$, if $p(a) = 0$, and $(ii)$ $p(a) = 0$, if $x - a$ is a factor of $p(x)$.*

*Proof.* By remainder Theorem, for any polynomial $p(x) \exists q(x)$ such that $p(x) = (x - a)q(x) + p(a)$. So

i If $p(a) = 0$, then $p(x) = (x - a)q(x)$, which show that $x - a$ is a factor of $p(x)$.

ii Since $x-a$ is a factor of $p(x)$, then $p(x) = (x-a)g(x)$ for some polynomial $g(x)$. So $p(a) = (a-a)g(a) = 0$.

□

**Example 2.7.2.** Examine whether $x+2$ is a factor of $x^3+3x^2+5x+6$ and $2x+4$.

**Solution** Now the zero of polynomial $x+2$ is $-2$, $p(x) = x+2 \implies -2+2=0$. We have polynomial $p(x) = x^3+3x^2+5x+6$, then

$$p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6 = -8 + 12 - 10 + 6 = 18 - 18 = 0$$

So, by factor teheorem,$x+2$ is a factor of $x^3+3x^2+5x+6$. Next consider $g(x) = 2x+4$,

$$g(-2) = 2(-2) + 4 = -4 + 4 = 0$$

So, by factor teheorem, $x+2$ is a factor of $2x+4$.

**Example 2.7.3.** Find the value of $k$, if $x-1$ is a factor of $4x^3+3x^2-4x+k$.

**Solution** If the monomial $x-1$ is a factor of $p(x) = 4x^3+3x^2-4x+k$, then by factor theorem $p(1) = 0$.

$$p(1) = 0 \implies 4(1)^3+3(1)^2-4(1)+k = 0 \implies 4+3-4+k = 0 \implies k = -3$$

## 2.8 Solution of exercise 2.3

**Problem 2.8.1.** Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

i $x + 1$

ii $x - \frac{1}{2}$

iii $x$

iv $x + \pi$

v $5 + 2x$

**Solution:** Using Remainder Theorem

i We have $p(x) = x^3 + 3x^2 + 3x + 1$, putting $x = -1$. We get remainder

$$p(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = -1 + 3 - 3 + 1 = 0$$

ii We have $p(x) = x^3 + 3x^2 + 3x + 1$, putting $x = \frac{1}{2}$. We get remainder

$$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 1 = \frac{1}{8} + 3\frac{1}{4} + 3\frac{1}{2} + 1 = \frac{63}{8}$$

iii We have $p(x) = x^3 + 3x^2 + 3x + 1$, putting $x = 0$. We get remainder

$$p(0) = (0)^3 + 3(0)^2 + 3(0) + 1 = 0 + 0 + 0 + 1 = 1$$

iv We have $p(x) = x^3 + 3x^2 + 3x + 1$, putting $x = -\pi$. We get remainder

$$p(-\pi) = (-1pi)^3 + 3(-\pi)^2 + 3(-\pi) + 1 = -\pi + 3\pi^2 - 3\pi + 1$$

v We have $p(x) = x^3 + 3x^2 + 3x + 1$, putting $x = -\frac{5}{2}$. We get remainder

$$p\left(-\frac{5}{2}\right) = \left(-\frac{5}{2}\right)^3 + 3\left(-\frac{5}{2}\right)^2 + 3\left(-\frac{5}{2}\right) + 1 = -\frac{125}{8} + 3\frac{25}{4} - 3\frac{5}{2} + 1 = -\frac{87}{8}$$

**Problem 2.8.2.** Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.

**Solution:** We have polynomial $p(x) = x^3 - ax^2 + 6x - a$, when $p(x)$ is divided by $x - a$ then by Remainder Theorem, remainder get

$$p(a) = a^3 - aa^2 + 6a - a = a^3 - a^3 + 5a = 5a$$

**Problem 2.8.3.** Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

**Solution:** We have polynomial $p(x) = 3x^3 + 7x$, then by Remainder Theorem the remainder when $p(x)$ is divided by $7 + 3x$ is $p\left(-\frac{7}{3}\right)$.

$$p\left(-\frac{7}{3}\right) = 3\left(-\frac{7}{3}\right)^3 + 7\left(-\frac{7}{3}\right) = -\frac{490}{9}.$$

So by Factor Theorem the $7 + 3x$ is not factor of $p(x) = 3x^3 + 7x$ because remainder is not 0.

### Factorisation of polynomial by splitting the middle term

Let quadratic is $ax^2 + bx + c$. To factorise the quadratic follow the following steps,

i Multiply the coefficien of $x^2$ and constant. Let $m = a \times c$.

ii Factorise the $m$ as possible way. let possible factor of $m$ are $a \times c, r \times s, p \times q$.

iii Consider that factor of $m$ which sum is equal to the ceofficient of $x$, as $b = r + s$.

iv Now write the quadratic as $ax^2 + (r + s)x + c$.

After simplification we get factor of quadratic.

**Example 2.8.4.** Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem.

**Solution 1:** (By splitting method): The given quadratic is $6x^2 + 17x + 5$, now compairing to general form of quadratic polynomial we get $a = 6$, $b = 17$, $c = 5$. So $m = a \times c = 6 \times 5 = 30$. Now need find two numbers $p$, $q$ such that $p + q = 17$.

| $m = p \times q$ | $p + q = b$ | $p + q = 17$ | *choice* |
|---|---|---|---|
| $1 \times 30$ | $1 + 30$ | 31 | |
| $2 \times 15$ | $2 + 15$ | 17 | ✓ |
| $3 \times 10$ | $3 + 10$ | 13 | |
| $5 \times 6$ | $5 + 6$ | 11 | |

From the table,

$$6x^2 + 17x + 5 = 6x^2 + (2 + 15)x + 5$$

$$= 6x^2 + 2x + 15x + 5 = 2x(3x + 1) + 5(3x + 1)$$

$$= (3x + 1)(2x + 1)$$

**Solution 2** (By using the Factor Theorem): We have $6x^2 + 17x + 5 = 6\left(x^2 + \frac{17}{6} + \frac{5}{6}\right) = 6p(x)$, say. If $a$ and $b$ are the zeroes of $p(x)$, then we can write $6x^2 + 17x + 5 = 6(x - a)(x - b) = 6[x^2 - (a + b)x + ab]$. So $ab = \frac{5}{6}$ and $a + b = -\frac{17}{6}$. Let us look at some posibilities for $a$ and $b$.

| $a$ | $b$ | $p(a)$ | $p(b)$ |
|---|---|---|---|
| $-\frac{1}{2}$ | $\frac{1}{2}$ | $p\left(-\frac{1}{2}\right) \neq 0$ | $p\left(\frac{1}{2}\right) \neq 0$ |
| $-\frac{1}{3}$ | $\frac{1}{3}$ | $p\left(-\frac{1}{3}\right) = 0, \checkmark$ | $p\left(\frac{1}{3}\right) \neq 0$ |
| $\frac{5}{3}$ | $-\frac{5}{3}$ | $p\left(\frac{5}{2}\right) \neq 0$ | $p\left(-\frac{5}{2}\right) = 0, \checkmark$ |
| $\frac{5}{2}$ | $-\frac{5}{2}$ | $p\left(\frac{5}{2}\right) \neq 0$ | $p\left(-\frac{5}{2}\right) \neq 0$ |

Therefore $\left(x + \frac{1}{3}\right)$ and $\left(x + \frac{5}{2}\right)$ are factor of $p(x)$. So,

$$6x^2+17x+5 = 6\left(x + \frac{1}{3}\right)\left(x + \frac{5}{2}\right) = 6\left(\frac{3x + 1}{3}\right)\left(\frac{2x + 5}{2}\right) = (3x+1)(2x+5).$$

**Example 2.8.5.** Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

**Solution:** Let $p(y) = y^2 - 5y + 6$. Now, if $p(y) = (y - a)(y - b)$, you know that the constant term will be $ab$. So, $ab = 6$. Therefore, to determine of the factor $p(y)$, we determine the factor 6, because any one factor of 6 will be $a$ and $b$. The factor of 6 are $1, 2$ and 3.

Now, $p(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$.

So, $y - 2$ is a factor of $p(y)$.

Next $p(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$.

So, $y - 3$ is also a factor of $p(y)$.

Therefore, $y^2 - 5y + 6 = (y - 2)(y - 3)$.

**Example 2.8.6.** Factorise $x^3 - 23x^2 + 142x - 120$.

**Solution:** Let $p(x) = x^3 - 23x^2 + 142x - 120$. If $p(x) = (x-a)(x-b)(x-c)$, then $abc = -120$, when coefficiend of $x^3$ is 1. Now we shall look all the factor of $-120$, whics are the $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm20, \pm24, \pm30, \pm60$.

by trial metho, we find $p(1) = 1^3 - 23 \times 1^2 + 142 \times 1 - 120 = 1 - 23 + 142 - 120 = 0$.

So, $x - 1$ is factor of $p(x)$.

Similarally $p(10) = 10^3 - 23 \times 10^2 + 142 \times 10 - 120 = 1000 - 2300 + 1420 - 120 = 0$.

So, $x - 10$ will be the factor of $p(x)$.

Similarally $p(12) = 12^2 - 23 \times 12^2 + 142 \times 12 - 120 = 1728 - 3312 + 1704 - 120 = 0$.

So, $x - 12$ will be the factor of $p(x)$. Hence $p(x) = (x - 1)(x - 10)(x - 12)$.

## 2.9 Solution of Exercise 2.4

**Problem 2.9.1.** Determine which of the following polynomial has $(x + 1)$ factor:

i $x^3 + x^2 + x + 1$

ii $x^4 + x^3 + x^2 + x + 1$

iii $x^4 + 3x^3 + 3x^2 + x + 1$

iv $x^3 - x^2 - \left(2 + \sqrt{2}\right) x + \sqrt{2}$

**Solution:** If $x + 1$ is the factor of a polynomial $p(x)$, then $p(-1) = 0$.

i Let $p(x) = x^3 + x^2 + x + 1$ Now,

$$p(-1) = (-1)^3 + (-1)^2 + (-1) + 1 = -1 + 1 - 1 + 1 = 0.$$

So, $x + 1$ is the factor of $p(x)$.

ii Let $p(x) = x^4 + x^3 + x^2 + x + 1$.

Then,

$$p(-1) = (-1)^4 + (-1)^3 + (-1)^2 + (-1) + 1 = 1 - 1 + 1 - 1 + 1 = 1.$$

So, $x + 1$ is not the factor of $x^3 + x^2 + x + 1$.

iii Let $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$.

Then,

$$p(-1) = (-1)^4 + 3(-1)^3 + 3(-1)^2 + (-1) + 1 = 1 - 3 + 3 - 1 + 1 = 1.$$

So, $x + 1$ is not the factor of $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$.

iv Let $p(x) = x^3 - x^2 - \left(2 + \sqrt{2}\right) x + \sqrt{2}$.

Then,

$$p(-1) = (-1)^3 - (-1)^2 - \left(2 + \sqrt{2}\right)(-1) + \sqrt{2} = -1 - 1 + \left(2 + \sqrt{2}\right) + \sqrt{2}$$

$$= 2\sqrt{2}.$$

So, $x+1$ is not factor of the $p(x) = x^3 - x^2 - \left(2+\sqrt{2}\right)x + \sqrt{2}$.

**Problem 2.9.2.** Use Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

i $p(x) = 2x^3 + x^2 - 2x - 1$, $g(x) = x + 1$

ii $p(x) = x^3 + 3x^2 + 3x + 1$, $g(x) = x + 2$

iii $p(x) = x^3 - 4x^2 + x + 6$, $g(x) = x - 3$

**Solution:**

i Let $p(x) = 2x^3 + x^2 - 2x - 1$, and $g(x) = x + 1$, then

$$p(-1) = 2(-)^3 + (-1)^2 - 2(-1) - 1 = -2 + 1 + 2 - 1 = 0.$$

So, by Factor Theorem $g(x) = x + 1$ is a factor of $p(x)$.

ii Let $p(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x + 2$ then

$$p(-2) = (-2)^3 + 3(-2)^2 + 3(-2) + 1 = -8 + 3 \times 4 - 6 + 1$$

$$= -8 + 12 - 6 + 1 = -1.$$

So, by Factor Theorem $g(x)$ is not a factor of $p(x)$.

iii $p(x) = x^3 - 4x^2 + x + 6$ and $g(x) = x - 3$, then

$$p(3) = 3^3 - 4 \times 3^2 + 3 + 6 = 27 - 36 + 3 + 6 = 0.$$

So, by Factor Theorem $g(x)$ is a factor of $p(x)$.

**Problem 2.9.3.** Find the value of $k$, if $x - 1$ is a factor of $p(x)$ in each of the following cases:

i $p(x) = x^2 + x + k$

ii $p(x) = 2x^2 + kx + \sqrt{2}$

iii $p(x) = kx^2 - \sqrt{2}x + 1$

iv $p(x) = kx^2 - 3x + k$

**Solution:**

i Let $p(x) = x^2 + x + k$, if $x - 1$ is a factor of $p(x)$. Then,

$$p(1) = 0 \implies 1^2 + 1 + k = 0 \implies 2 + k = 0 \implies k = -2.$$

ii Let $p(x) = 2x^2 + kx + \sqrt{2}$, if $x - 1$ is a factor of $p(x)$. Then,

$$p(1) = 0 \implies 2.1^2 + k1 + \sqrt{2} = 0 \implies 2 + k + \sqrt{2} = 0 \implies k = -2 - \sqrt{2}.$$

iii Let $p(x) = kx^2 - \sqrt{2}x + 1$, if $x - 1$ is a factor of $p(x)$. Then,

$$p(1) = 0 \implies k.1 - \sqrt{2}.1 + 1 = k - \sqrt{2} + 1 = 0 \implies k = -1 + \sqrt{2}.$$

iv $p(x) = kx^2 - 3x + k$, if $x - 1$ is a factor of $p(x)$. Then,

$$p(1) = 0 \implies k.1^2 - 3.1 + k = 0 \implies k - 3 + 1 = 0 \implies 2k = 3 \implies k = \frac{3}{2}.$$

**Problem 2.9.4.** Factorise:

i $12x^2 - 7x + 1$

ii $2x^2 + 7x + 3$

iii $6x^2 + 5x - 6$

iv $3x^2 - x - 4$

**Solution:**

i Let $12x^2 - 7x + 1$ is given polynomial, then

$$12x^2 - 7x + 1 = 12x^2 - (4 + 3)x + 1$$

$$= 12x^2 - 4x - 3x + 1 = 4x(3x - 1) - 1(3x - 1)$$

$$= (3x - 1)(4x - 1).$$

ii Let $2x^2 + 7x + 3$ is a given polynomial, then

$$2x^2 + 7x + 3 = 2x^2 + (6 + 1)x + 3$$

$$= 2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3) = (x + 3)(2x + 1).$$

iii Let $6x^2 + 5x - 6$ is a given polynomial, then

$$6x^2 + 5x - 6 = 6x^2 + (9 - 4)x - 6$$

$$= 6x^2 + 9x - 4x - 6 = 3x(2x + 3) - 2(3x + 2)$$

$$= (2x + 3)(3x - 2).$$

iv Let $3x^2 - x - 4$ is a polynomial, then

$$3x^2 - x - 4 = 3x^2 + (3 - 4)x - 4$$

$$= 3x^2 + 3x - 4x - 4 = 3x(x + 1) - 4(x + 1)$$

$$= (x + 1)(3x - 4).$$

**Problem 2.9.5.** Factorise :

i $x^3 - 2x^2 - x + 2$

ii $x^3 - 3x^2 - 9x - 5$

iii $x^3 + 13x^2 + 32x + 20$

iv $2y^3 + y^2 - 2y - 1$

**Solution:**

i Let $x^3-2x^2-x+2$ is a polynomial. The constant term of polynamial is 2, now the factor of constant term are $\pm 1, \pm 2$. The value of polynomial at factor of constant term,

$$p(1) = 1^3 - 2 \times 1^2 - 1 + 2 = 1 - 2 - 1 + 2 = 0.$$

So, by Factor Theorem $x-1$ is factor of$x^3 - 2x^2 - x + 2$. Hence

$$x^3 - 2x^2 - x + 2 = x^2(x-1) - x(x-1) - 2(x-1) = (x-1)(x^2 - x - 2)$$

$$= (x-1)\left\{x^2 - 2x + x - 2\right\} = (x-1)\left\{x(x-2) + (x-2)\right\}$$

$$= (x-1)(x-2)(x+1).$$

ii Let $p(x) = x^3 - 3x^2 - 9x - 5$. Now the factors of constant term of $p(x)$ are $\pm 1, \pm 5$. Next

$$p(5) = 5^3 - 3 \times 5^2 - 9 \times 5 - 5 = 125 - 75 - 45 - 5 = 0.$$

So, by Factor Theorem $x-5$ is the factor of $p(x)$. Hence

$$x^3 - 3x^2 - 9x - 5 = x^2(x-5) + 2x^2 - 9x - 5 = x^2(x-5) + 2x(x-5) + x - 5$$

$$= (x-5)(x^2 + 2x + 1) = (x-5)(x+1)(x+1).$$

iii Let $p(x) = x^3 + 13x^2 + 32x + 20$. The factors of constant term of $p(x)$ are $\pm 2, \pm 5$. Now

$$p(-2) = (-2)^3 + 13 \times (-2)^2 + 32(-2) + 20 = -8 + 52 - 64 + 20 = 0.$$

So, by Factor Theorem $x+2$ is the factor of $p(x)$. Hence

$$p(x) = x^3 + 13x^2 + 32x + 20 = x^2(x+2) + 11x^2(x+2) + 10(x+2)$$

$$= (x+2)(x^2+11x+10) = (x+2)(x^2+x+10x+10) = (x+2)(x+1)(x+10).$$

iv Let $p(y) = 2y^3 + y^2 - 2y - 1$. The factor of constant term are $\pm 1$. Then

$$p(1) = 2.1^3 + 1^2 - 2 \times 1 - 1 = 2 + 1 - 2 - 1 = 0.$$

So, by Factor Theorem $y-1$ is the factor of $p(y)$. Hence

$$2y^3+y^2-2y-1 = 2y^2(y-1)+3y(y-1)+(y-1) = (y-1)(2y^2+3y+1)$$

$$= (y-1)(2y^2+2y+y+1) = (y-1)(y+1)(2y+1).$$

## 2.10 Algebraic Identities

**Algebraic Identity :** An algebraic identity is an algebraic equation that is true for all values of the variables occurring in it.

There are some algebraic identities,

**Identity I :** $(x+y)^2 = x^2 + 2xy + y^2$

**Identity II :** $(x-y)^2 = x^2 - 2xy + y^2$

**Identity III :** $x^2 - y^2 = (x+y)(x-y)$

**Identity IV :** $(x+a)(x+b) = x^2 + (a+b)x + ab$

**Example 2.10.1.** Find the following product using appropriate identities:

i $(x+3)(x+3)$

ii $(x-3)(x+5)$

**Solution :**

i We have identity I : $(x+y)^2 = x^2 + 2xy + y^2$. Putting $y = 3$ in identity.

We get

$$(x+3)(x+3) = (x+3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 3x + 9.$$

ii Using identity IV : $(x+a)(x+b) = x^2 + (a+b)x + ab$, we get

$$(x-3)(x+5) = x^2 + (-3+5)x + (-3)(5) = x^2 + 2x - 15.$$

**Example 2.10.2.** Evaluate $105 \times 106$ without multiplying directly.

**Solution :**

$$105 \times 106 = (100 = 5)(100 + 6)$$

$$= (100)^2 + (5+6)100 + 5 \times 6 = 10000 + 1100 + 30$$

$$= 11130.$$

**Example 2.10.3.** Factorise :

i $49a^2 + 70ab + 25b^2$

ii $\frac{25}{4}x^2 - \frac{y^2}{9}$

**Solution :** Now we can write the

i $49a^2 + 70ab + 25b^2$, as

$$49a^2 + 70ab + 25b^2 = (7a)^2 + 2(7a)(5b) + (5b)^2$$

$$= (7a + 5b)^2 = (7a + 5b)(7a + 5b)$$

ii We have $\frac{25}{4}x^2 - \frac{y^2}{9}$. It can be write

$$\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2$$

Using the identity III, we get

$$\left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2 = \left(\frac{5}{2}x - \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right)$$

**Identity V :** $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

**Identity VI :** $(x + y)^3 = x^3 + y^3 + 3xy(x + y) = x^3 + y^3 + 3x^2y + 3xy^2$

**Identity VII :** $(x - y)^3 = x^3 - y^3 - 3xy(x - y) = x^3 - y^3 = 3x^2y + 3xy^2$

**Example 2.10.4.** Write $(3a + 4b + 5c)^2$ in expanded form.

**Solution :** Comparing the given expression $(3a+4b+5c)^2$ with $(x+y+z)^2$, we find that

$$x = 3a, y = 4b, z = 5c.$$

So, using Identiti V, we get

$$(3a+4b+5c)^2 = (3a)^2+(4b)^2+(5c)^2+2(3a)(4b)+2(4b)(5c)+2(5c)(3a)$$

$$= 9a^2+16b^2+25c^2+24ab+40bc+30ca$$

**Example 2.10.5.** Expand $(4a-2b-3c)^2$.

**Solution :** We have $(4a-2b-3c)^2$, now using Identity V,

$$(4a-2b-3c)^2 = [4a+(-2b)+(-3c)]^2$$

$$= (4a)^2+(-2b)^2+(-3c)^2+2(4a)(-2b)+2(-2b)(-3c)+2(-3c)(4a)$$

$$= 16a^2+4b^2+9c^2-16ab+12bc-24ac$$

**Example 2.10.6.** Factorise $4x^2+y^2+z^2-4xy-2yz+4xz$.

**Solution :** We have expression $4x^2+y^2+z^2-4xy-2yz+4xz$. Now it can be express as

$$4x^2+y^2+z^2-4xy-2yz+4xz = (2x)^2+(-y)^2+(z)^2+2(2x)(-y)+2(-y)(z)+2(2x)(z)$$

using identity V

$$= [2x-y+z]^2 = (2x-y+z)(2x-y+z)$$

**Example 2.10.7.** Wirte the following cubes in the expanded form:

i $(3a+4b)^3$

ii $(5a - 3b)^3$

**Solution :**

i Using identity $VI$

$(3a + 4b)^3 = (3a)^3 + (4b)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 = 27a^3 + 64b^3 + 108a^2b + 144ab^2$.

ii Using identity $VI$

$(5a - 3b)^3 = (5a^3) + (-3b)^3 + 3(5a)^2(-3b) + 3(5a)(-3b)^2 = 125a^3 - 27b^3 - 225a^2b + 135ab^2$.

Use of Identity for numerical calculation

**Example 2.10.8.** Evaluate each of the following using sutable identies:

i $(104)^3$

ii $(999)^3$

**Solution:**

i We have

$$(104)^3 = (100 + 4)^3$$

$$= 100^3 + 4^3 + 3 \times 100^2 \times 4 + 3 \times 100 \times 4^2$$

$$= 1000000 + 16 + 12 \times 10000 + 48 \times 100$$

$$= 1000000 + 16 + 120000 + 4800 = 101204816.$$

ii We have

$$(999)^3 = (1000 - 1)^3$$

$$= 1000^3 - 1^3 + 3 \times 1000^2 \times (-1) + 3 \times 1000 \times (-1)^2$$

$$= 1000000000 - 1 - 3 \times 1000000 + 3 \times 1000$$

$$= 1000000000 - 1 - 3000000 + 3000 = 1000003000 - 3000001$$

$$= 99702999.$$

**Example 2.10.9.** Factorise $8x^3 + 27y^3 + 36x^2 + 54xy^2$

**Solution:** The given expression can be written as,

$$8x^3 + 27y^3 + 36x^2 + 54xy^2 = (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y^2)$$

$$= (2x + 3y)^3 = (2x + 3y)(2x + 3y)(2x + 3y)$$

**Identity $VIII$:**

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + x^2 - xy - yz - zx)$$

**Example 2.10.10.** Factorise : $8x^3 + y^3 + 27z^3 - 18xyz$

**Solution :** We have

$$8x^3 + y^3 + 27z^3 - 18xyz = (2x)^3 + (y^3) + (3z)^3 - 3(2x)(y)(3z)$$

$$= (2x + y + 3z)\left[(2x)^2 + y^2 + (3z)^2 - (2x)y + y(3z) - (3z)(2x)\right]$$

$$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6zx)$$

using identity $VIII$

## 2.11 Solution of Exercise 2.5

**Problem 2.11.1.** Use sutaible identities to find the following products:

i $(x + 4)(x + 10)$

ii $(x + 8)(x - 10)$

iii $(3x + 4)(3x - 5)$

iv $\left(y^2 + \frac{3}{2}\right)\left(y^2 - \frac{3}{2}\right)$

v $(3 - 2x)(3 + 2x)$

**Solution:**

i Using identities $IV$, we get

$$(x + 4)(x + 10) = x^2 + (4 + 10)x + 4 \times 10 = x^2 + 14x + 40$$

ii Using identities $IV$, we get

$$(x + 8)(x - 10) = x^2 + (8 - 10)x + 8 \times (-10) = x^2 - 2x - 80$$

iii Using identities $IV$, we get

$$(3x+4)(3x-5) = (3x)^2 + (4-5)(3x) + 4(-5) = 9x^2 - 3x - 20$$

iv Using identities $III$, we get

$$\left(y^2 + \frac{3}{2}\right)\left(y^2 - \frac{3}{2}\right) = \left(y^2\right)^2 - \left(\frac{3}{2}\right)^2 = y^4 - \frac{9}{4}$$

v Using identities $III$, we get

$$(3-2x)(3+2x) = 3^2 - (2x)^2 = 9 - 4x^2$$

**Problem 2.11.2.** Evaluate the following products without multiplying directly:

i $103 \times 107$

ii $95 \times 96$

iii $104 \times 96$

**Solution:** Using Identity $IV$

i It cab be written as $103 \times 107 = (100+3)(100+7)$

$$(100+3)(100+7) = 100^2 + (3+7)100 + 3 \times 7 = 10000 + 10 \times 100 + 21$$

$$= 10000 + 1000 + 21 = 11021$$

ii It cab be written as $95 \times 96 = (100 - 5)(100 - 4)$

$$(100-5)(100-4) = 100^2+(-5-4)100+(-5)(-4) = 10000-9\times100+20$$

$$= 10000 - 900 + 20 = 9120$$

iii It cab be written as $104 \times 96 = (100 + 4)(100 - 4)$ Using Identitiy $III$

$$(100 + 4)(100 - 4) = 100^2 - 4^2 = 10000 - 16 = 9984$$

**Problem 2.11.3.** Factorise the following using appropriate identities:

i $9x^2 + 6xy + y^2$

ii $4y^2 - 4y + 1$

iii $x^2 - \frac{y^2}{100}$

**Solution:**

i It can be written as

$$9x^2 + 6xy + y^2 = (3x)^2 + 2 \times 3x \times y + y^2$$

Using Identity $I$

$$(3x)^2 + 2 \times 3x \times y + y^2 = (3x + y)^2 = (3x + y)(3x + y)$$

ii It can be written as

$$4y^2 - 4y + 1 = (2y)^2 - 2 \times 2y \times 1 + 1^2$$

Using Identity $II$

$$(2x)^2 - 2 \times 2x \times 1 + 1^2 = (2y-1)^2 = (2y-1)(2y-1).$$

iii It can be written as

$$x^2 - \frac{y^2}{100} = (x)^2 - \left(\frac{y}{10}\right)^2$$

Using Identity $III$

$$(x)^2 - \left(\frac{y}{10}\right)^2 = \left(x + \frac{y}{10}\right)\left(x - \frac{y}{10}\right)$$

**Problem 2.11.4.** Expand each of the following, using suitable identities:

i $(x + 2y + 4z)^2$

ii $(2x - y + z)^2$

iii $(-2x + 3y + 2z)^2$

iv $(3a - 7b - c)^2$

v $(-2x + 5y - 3z)^2$

vi $\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2$

**Solution:** Using the Identity $V$

i The given expression $(x+2y+4z)^2$, so

$$(x+2y+4z)^2 = x^2+(2y)^2+(4z)^2+2x(2y)+2(2y)(4z)+2(4z)x$$

$$= x^2+4y^2+16z^2+4xy+16yz+8zx$$

ii The given expression $(2x-y+z)^2$, so

$$(2x-y+z)^2 = (2x)^2+(-y)^2+z^2+2(2x)(-y)+2(-y)z+2z(2x)$$

$$= 4x^2+y^2+z^2-4xy-2yz+2zx$$

iii The given expression $(-2x+3y+2z)^2$, so

$$(-2x+3y+2z)^2 = (-2x)^2+(3y)^2+(2z)^2+2(-2x)(3y)+2(3y)(2z)+2(2z)(-2x)$$

$$= 4x^2+9y^2+4z^2-12xy+12yz-8zx$$

iv The given expression $(3a-7b-c)^2$, so

$$(3a-7b-c)^2 = (3a)^2+(-7b)^2+(-c)^2+2(3a)(-7b)+2(-7b)(-c)+2(-c)(3a)$$

$$= 9a^2+49b^2+c^2-42ab+14bc-6ca$$

v The given expression $(-2x+5y-3z)^2$, so

$$(-2x+5y-3z)^2 = (-2x)^2+(5y)^2+(-3z)^2+2(-2x)(5y)+2(5y)(-3z)+2(-3z)(-2x)$$

$$= 4x^2+25y^2+9z^2-20xy-30yz+12zx$$

vi The given expression $\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2$, so

$$\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2 = \left(\frac{1}{4}a\right)^2 + \left(-\frac{1}{2}b\right) + 1^2 + 2\left(\frac{1}{4}a\right)\left(-\frac{1}{2}b\right) + 2\left(-\frac{1}{2}b\right)\times 1 + 2\times 1\left(\frac{1}{4}a\right)$$
$$= \frac{1}{16}a^2 + \frac{1}{4}b^2 + 1 - \frac{1}{4}ab - b + \frac{1}{2}a$$

**Problem 2.11.5.** Factorise:

i $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16zx$

ii $2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy + 4\sqrt{2}yz - 8zx$

**Solution:** Using identity $V$

i We have expression $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16zx$. It can be written as

$$4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16zx$$
$$= (2x)^2 + (3y)^2 + (-4z)^2 + 2(2x)(3y) + 2(3y)(-4z) + 2(-4z)(2x)$$
$$= (2x + 3y - 4z)^2 = (2x + 3y - 4z)(2x + 3y - 4z)$$

ii We have expression $2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy + 4\sqrt{2}yz - 8zx$. So it can be written as

$$2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy + 4\sqrt{2}yz - 8zx$$
$$= \left(-\sqrt{2}x\right)^2 + y^2 + \left(2\sqrt{2}z\right)^2 + 2\left(-\sqrt{2}x\right)y + 2y\left(2\sqrt{2}z\right) + 2\left(2\sqrt{2}z\right)\left(-\sqrt{2}x\right)$$
$$= \left(-\sqrt{2}x + y + 2\sqrt{2}z\right)^2 = \left(-\sqrt{2}x + y + 2\sqrt{2}z\right)\left(-\sqrt{2}x + y + 2\sqrt{2}z\right)$$

**Problem 2.11.6.** Write the following cubes in expanded form:

i $(2x+1)^3$

ii $(2a-3b)^3$

iii $\left[\frac{3}{2}x+1\right]^3$

iv $\left[x-\frac{2}{3}y\right]^3$

**Solution:** Using the Idenditi $VI$ for $i$ and $iii$, and identity $VII$ for $ii$ and $iv$,

i We have $(2x+1)^3$, so

$$(2x+1)^3 = (2x)^3 + 1^3 + 3(2x)^2 \times 1 + 3(2x)1^2 = 8x^3 + 1 + 12x^2 + 6x$$

ii We have $(2a-3b)^3$, so

$$(2a-3b)^3 = (2a)^3 - (3b)^3 - 3(2a)^2(3b) + 3(2a)(3b)^2$$

$$= 8a^3 - 27b^3 - 36a^2b + 54ab^2$$

iii We have $\left[\frac{3}{2}x+1\right]^3$, so

$$\left[\frac{3}{2}x+1\right]^3 = \left(\frac{3}{2}x\right)^3 + 1^3 + 3\left(\frac{3}{2}x\right)^2 \times 1 + 3\left(\frac{3}{2}x\right)1^2$$

$$= \frac{27}{8}x^3 + 1 + \frac{27}{4}x^2 + \frac{9}{2}x$$

iv We have $\left[x - \frac{2}{3}y\right]^3$, so

$$\left[x - \frac{2}{3}y\right]^3 = x^3 - \left(\frac{2}{3}y\right)^3 - 3x^2\left(\frac{2}{3}y\right) + 3x\left(\frac{2}{3}y\right)^2$$

$$= x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4}{3}y^2x$$

**Problem 2.11.7.** Evaluate the following using sutable identites:

i $(99)^3$

ii $(102)^3$

iii $(998)^3$

**Solution:** To evaluate the problem $i$ and $iii$ apply the identities $VII$ and for $ii$ apply identities $VI$.

i It can be express as $(99)^3 = (100 - 1)^3$. So,

$$(100 - 1)^3 = 100^3 - 1^3 - 3 \times 100^2 \times 1 + 3 \times 100 \times 1$$

$$1000000 - 1 - 30000 + 300 = 970299.$$

ii It can be express as $(102)^3 = (100 + 2)^3$. So,

$$(100 + 2)^3 = 100^3 + 2^3 + 3 \times 100^2 \times 2 + 3 \times 100 \times 2^2$$

$$= 1000000 + 8 + 3 \times 10000 \times 2 + 3 \times 100 \times 4 = 1000000 + 8 + 60000 + 1200$$

$$= 1061208$$

iii It can be express as $(998)^3 = (1000-2)^3$. So,

$$(1000-2)^3 = 1000^3 - 2^3 - 3\times 1000^2 \times 2 + 3\times 1000\times 2^2$$

$$= 100000000 - 8 - 6\times 1000000 + 12000 = 994011992$$

**Problem 2.11.8.** Factorise each of the following:

i $8a^3 + b^3 + 12a^2b + 6ab^2$

ii $8a^3 - b^3 - 12a^2b + 6ab^2$

iii $27 - 125a^3 - 135a + 225a^2$

iv $64a^3 - 27b^3 - 144a^2b + 108ab^2$

v $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$

**Solution:**Apply the identity $VI$ for $i$ and $VII$ for (ii, iii, iv,v)

i We have expression $8a^3 + b^3 + 12a^2b + 6ab^2$.

$$8a^3 + b^3 + 12a^2b + 6ab^2 = (2a)^3 + b^3 + 3(2a)^2(b) + 3(2a)(b^2)$$

$$= (2a+b)^3 = (2a+b)(2a+b)(2a+b)$$

ii We have expression $8a^3 - b^3 - 12a^2b + 6ab^2$. S0,

$$8a^3 - b^3 - 12a^2b + 6ab^2 = (2a)^3 - b^3 - 3(2a)^2(b) + 3(2a)(b^2)$$

$$= (2a-b)^3 = (2a-b)(2a-b)(2a-b)$$

iii We have expression $27 - 125a^3 - 135a + 225a^2$. So,

$$27 - 125a^3 - 135a + 225a^2 = 3^3 - (5a)^3 + 3(5a)^2 3 - 3(5a)3^2$$

$$= (3 - 5a)^3 = (3 - 5a)(3 - 5a)(3 - 5a)$$

iv We have expression $64a^3 - 27b^3 - 144a^2b + 108ab^2$. So,

$$64a^3 - 27b^3 - 144a^2b + 108ab^2 = (4a)^3 - (3b)^3 - 3(4a)^2(3b) + 3(4a)(3b)^2$$

$$= (4a - 3b)^3 = (4a - 3b)(4a - 3b)(4a - 3b).$$

v We have expression $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$. So,

$$27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p = (3p)^3 - \left(\frac{1}{6}\right)^3 - 3(3p)^2\frac{1}{6} + 3\left(\frac{1}{6}\right)^2(3p)$$

$$= \left(3p - \frac{1}{6}\right)^3 = \left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right).$$

**Problem 2.11.9.** Verify:

i $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

ii $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

**Solution:**

i From equation $i$

$$RHS = (x + y)(x^2 - xy + y^2) = x^3 - x^2y + xy^2 + yx^2 - xy^2 + y^3$$

$$= x^3 + y^3 = LHS$$

ii From equation $ii$

$$RHS = (x - y)(x^2 + xy + y^2) = x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3$$

$$= x^3 - y^3 = LHS$$

**Problem 2.11.10.** Factorise each of the following:

i $27y^3 + 125z^3$

ii $64m^3 - 343n^3$

**Solution:**

i we have $27y^3 + 125z^3$. Using identity $VI$

$$27y^3 + 125z^3 = (3y + 5z)\left[(3y)^2 - (3y)(5z) + (5z)^2\right]$$

$$= (3y + 5z)(9y^2 - 15yz + 25z^2)$$

ii We have $64m^3 - 343n^3$. Using identity $VII$

$$64m^3 - 343n^3 = (4m - 7n)\left[(4m)^2 + (4m)(7n) + (7n)^2\right]$$

$$= (4m - 7n)(16m^2 + 28mn + 49n^2).$$

**Problem 2.11.11.** Factorise $27x^3 + y^3 + z^3 - 9xyz$.

**Solution:** Apply identity $VIII$, it can be express as

$$27x^3 + y^3 + z^3 - 9xyz = (3x + y + z)\left((3x)^2 + y^2 + z^2 - (3x)y - yz - z(3x)\right)$$

$$= (3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3zx)$$

**Problem 2.11.12.** Verify

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$$

**Solution:**

$$RHS = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$$

$$= \frac{1}{2}(x + y + z)\left[x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + z^2 - 2zx + x^2\right]$$

$$= \frac{1}{2}(x + y + z)(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$$

$$= \frac{1}{2} \times 2(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= x^3 + y^3 + z^3 - 3xyz = LHS$$

**Problem 2.11.13.** If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.

**Solution:** From identity $VIII$,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

, putting $x + y + z = 0$, We get

$$x^3 + y^3 + z^3 - 3xyz = 0 \times (x^2 + y^2 + z^2 - xy - yz - zx)$$

$$x^3 + y^3 + z^3 - 3xyz = 0$$

$$x^3 + y^3 + z^3 = 3xyz.$$

**Problem 2.11.14.** Without actually calculating the cubes, find the value of each of the following:

i $(-12)^3 + (7)^3 + (5)^3$

ii $(28)^3 + (-15)^3 + (-13)^3$

**Solution:**

i Here given expression $(-12)^3 + (7)^3 + (5)^3$ and $-12 + 5 + 7 = 0$, So use the result of problem (2.11.14), we get

$$(-12)^3 + (7)^3 + (5)^3 = 3 \times (-12) \times 5 \times 7 = -1260$$

ii Here given expression $(28)^3 + (-15)^3 + (-13)^3$ and $28 - 15 - 13 = 0$, So use the result of problem 2.11.14, we get

$$(28)^3 + (-15)^3 + (-13)^3 = 3 \times 28(-15)(-13) = 16380.$$

**Problem 2.11.15.** Give possible expressions for the length and breadth of each of the following rectangles, in which their area are given:

i Area: $25a^2 - 35a + 12$

ii Area: $35y^2 + 13y - 12$

**Solution:** The area of recatngle is the product of length and breadth, then two possible factor of area of rectangle will be length and breadth of given rectangle,

i Area of rectangle is $25a^2 - 36a + 12$. So,

$$25a^2 - 35a + 12 = 25a^2 - 20a - 15a + 12 = 5a(5a - 4) - 3(5a - 4)$$

$$= (5a - 4)(5a - 3)$$

Hence, Lenth=$(5a - 4)$ and Breadth=$(5a - 3)$

ii Area of rectangle is $35y^2 + 13y - 12$. So,

$$35y^2 + 13y - 12 = 35y^2 + 28y - 15y - 12 = 7y(5y + 4) - 3(5y + 4)$$

$$= (5y + 4)(7y - 3)$$

Hence, Length=$(5y + 4)$ and Breadth=$(7y - 3)$

**Problem 2.11.16.** What is the possible expressions for the dimensions of the cuboids whose volumes are given below ?

i Volume: $3x^2 - 12x$

ii Volume: $12ky^2 + 8ky - 20k$

**Solution:** The volume of cuboid $V = lbh$, So possible three factor of volume expression will be lenth, breadth and height of cuboid.

i Volume is $3x^2 - 12x$. So,

$$3x^2 - 12x = 3x(x - 4)$$

. Hence Length=3, Breadth=$x$ and Height=$(x - 4)$.

ii Volume is $12ky^2 + 8ky - 20k$. So,

$$12ky^2 + 8ky - 20k = 4k(3y^3 + 2y - 5) = 4k(3y^2 + 5y - 3y - 5)$$

$$= 4k\left[y(3y + 5) - (3y + 5)\right] = 4k(3y + 5)(y - 1)$$

Hence Length=$4k$, Breadth=$(3y + 5)$ and Height=$(y - 1)$.

# 3 COORDINATE GEOMETRY

## 3.1 Introduction

In number system you have seen that each real number can be placed a unique place on real line.

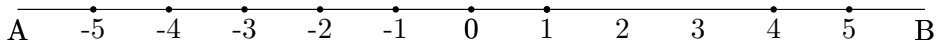



Figure 3.1.1: Positioning of point

But when we talk about position of a point in a plane then you can not decide the position of point by only one number line.

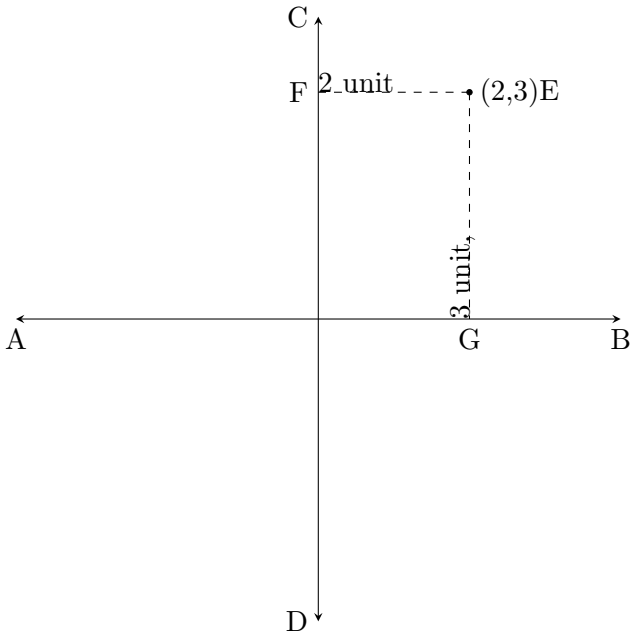



Figure 3.1.2: Positioning of point

Consider a point $E(2,3)$, how to find the correct position of 'E' ? We need two perpendicular lines $AB$ and $CD$. The position of point $E$ is 2 unit fram line $CD$ and 3 unit from $AB$, that is exact position of $E$.

**Note:** The distance of a point from a line means perpendicular distance of point from line.

## 3.2 Solution of Exercise 3.1

**Problem 3.2.1.** How you discribe the position of lamp on your study table to another person?

**Solution:** Frist we name the side of table, then help of any two perpendicular and adjacent side of table.

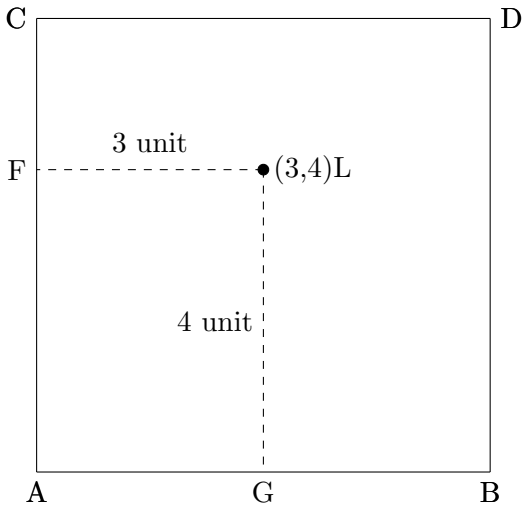



Figure 3.2.1: Positioning of point

Now see in Fig. (3.2.1), let ABCD the surface of table and the position of lamp is on point L. The position of L is 3-unit distance from line AC along AB and 4-unit distance from AB along to line AC.That is exact position of lamp.

**Problem 3.2.2. (Street Plan):** A city has two main roads which cross each other at the center of the city. These two roads are along the North-South direction and East-West direction.

All the other streets of the city run parallel to these roads and are 200 m apart. There are 5 streets in each direction. Using $1cm = 200\ m$, draw a model of the city on your note book. Represent the roads/streets by single line.

There are many cross-streets in your model. A particular cross-street is made by two-street, one running in the North-South direction another in the East-West direction. Each cross street is referred to in the following manner: If the $2^{nd}$ street running in the Nortn-South direction and $5^{th}$ in the East-West direction meet at some crossing, then we will call this cross-street $(2, 5)$. Using this convention, find:

i How many cross-streets can be referred to as $(4, 3)$.

ii How many cross-streets can be referred to as $(3, 4)$.

**Solution:** To draw model of city, let 1 $cm$ = 200 $m$, now draw the main roads which are North-South direction and other in East-West direction. All another 5 streets draw parallel to these two main streets.

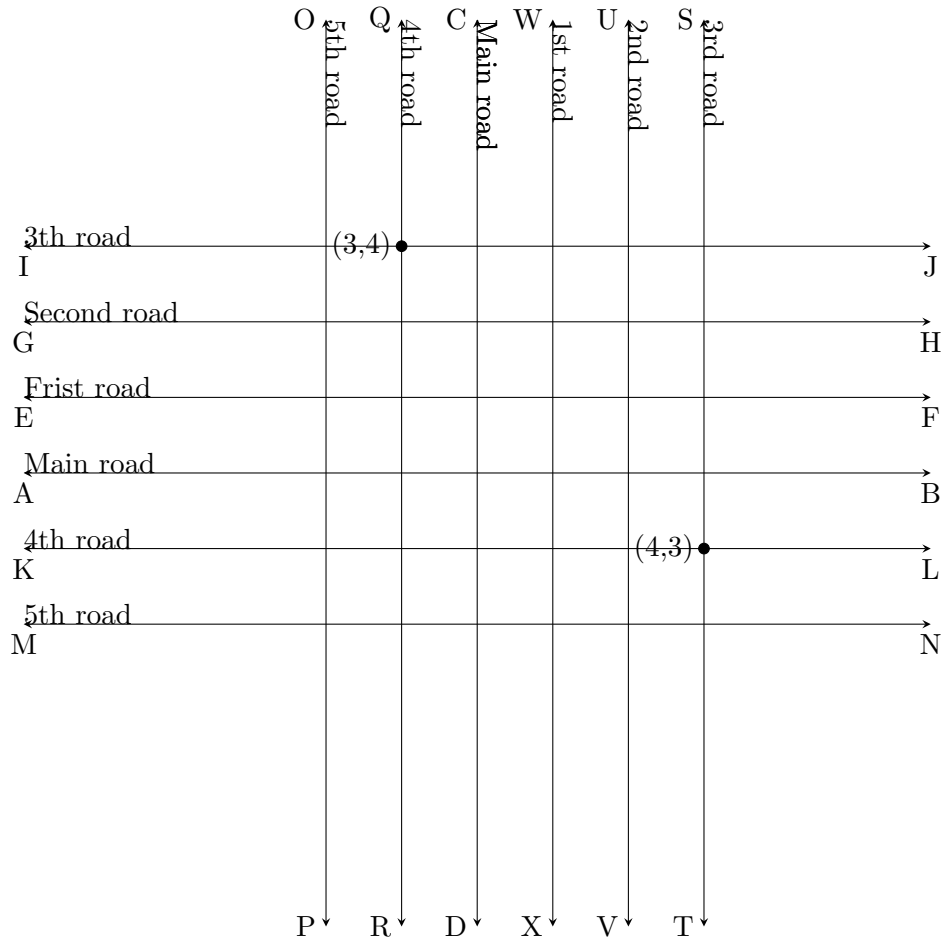



Figure 3.2.2: Model of city

i Only one cross-streets referred to $(4, 3)$.

ii Only one cross-streets can be referred to $(3, 4)$.

## 3.3 Cartesian System

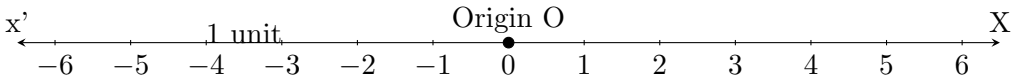



Figure 3.3.1: Ploting of point on horizontal line

We have been studied the number line in chapter one. In that chapter we learn that how to place a real number on real line. There are positive and negative real numbers which was correcly placed on real number line. Hence each real number associated to a distance, each distance measured from a fixed point. The fixed point from which distance mesured for a real number called **origin.** The distance measured left side of the origin or right side of the origin represents as negative or positive real number. Similar manner You can draw a verticle line as well as horizontao line.

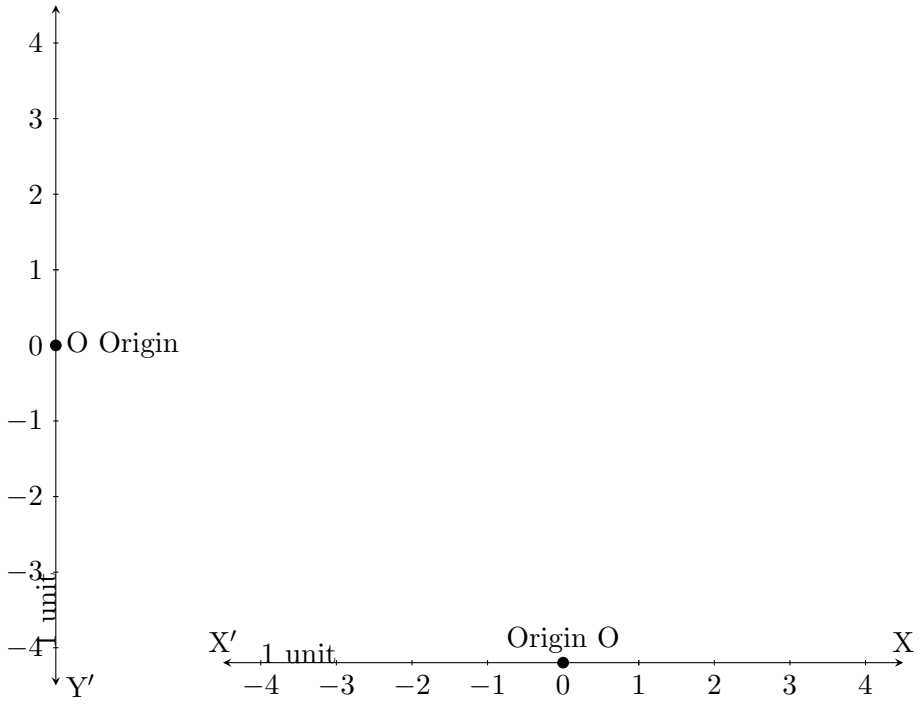



Figure 3.3.2: Ploting of point on vertical and horizontal line

When you combine these two horizontal and vertical line, you find a system that may help you to determine the position of point.

Now here we shall develop the system that determine exact position of point in a plane. There needs two lines, one is horizontal and other is verticle line that means two perpendicular line. These lines are called $X'X$ as $x$**-axis** and $Y'Y$ as $y$**-axis**. The intersection of $x$-axis and $y$-axis is called **origin**. The $OX$ and $OY$ are called the positive direction of $x-$axis and $y-$ axis. Similarly $OX'$ and $OY'$ are called the negative direction of $x-$axis and $y-$ axis. These

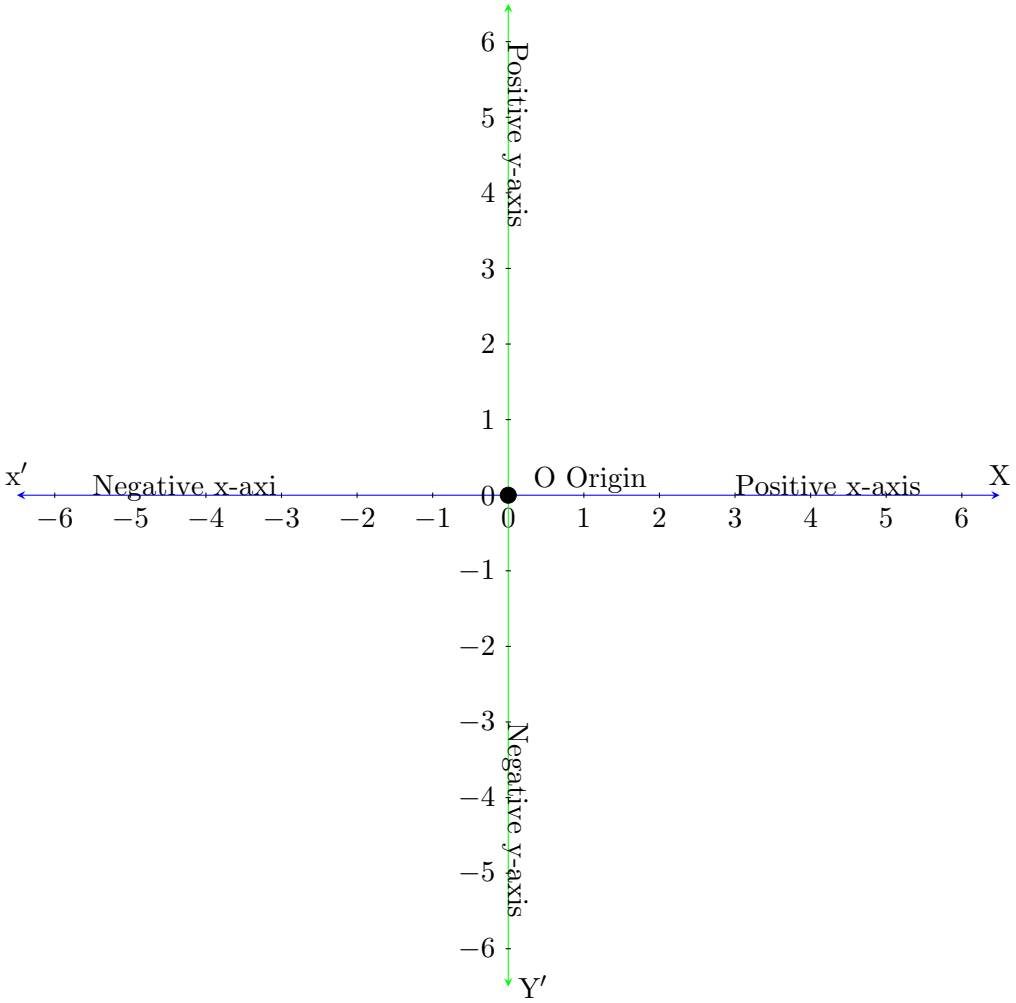



Figure 3.3.3: Coordinate axies

two lines devide the whole plane in four part, each part called $I$ quadrant, $II$ quadrant, $III$ quadrant, $IV$ quadrant.

To determine a point uniqualy in coordinate plane or $xy-$ plane. Define two co-ordinate associated to each point in the $xy-$ plane.

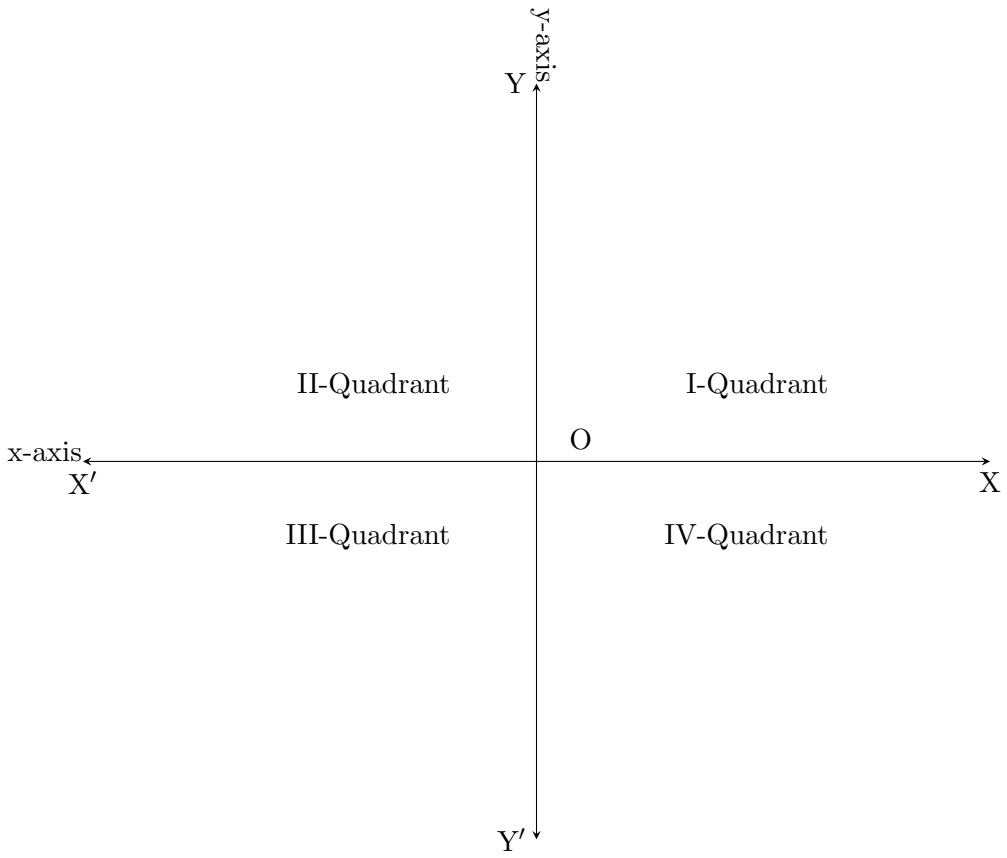



Figure 3.3.4: Coordinate System

**Rule to determine the coordinate of a point :**

i The $x-coordinate$ of a point is its perpendicular distance from $y-axis$.

The $x-coordinate$ is also called the **abscissa**.

ii The $y-coordinate$ of a point is its perpendicular distance from $x-axis$.

The $y-coordinate$ is also called the **ordinate**.

iii To writting the coordinate of a point in the coordinate plane, the $x-$ *coordinate* comes frist, and then the $y-$ *coordinate*. These coordinates place in small brackets.

**Example 3.3.1.** See the Fig.(3.3.5) and complete the following statemants:

i The abscissa and ordinate of the point B are ... and ..., respectively. Hence coordinate of B are (...,...).

ii The x-coordinate and y-coordinate of the point M are ... and ..., respectively. Hence coordinate of M are (...,...).

iii The x-coordinate and y-coordinate of the point L are ... and ..., respectively. Hence coordinate of L are (...,...).

iv The x-coordinate and y-coordinate of the point S are ... and ..., respectively. Hence coordinate of S are (...,...).

**Solution:**

i In Fig.(3.3.5) the distance of point B from the y-axis is 4 units, the x-coordinate or abscissa of the point B is 4. The distance of the point B from x-axis is 2.5 unit, the y-coordinate or ordinate of the point B is 2.5. Hence the coordinate of the point B are $(4, 2.5)$.

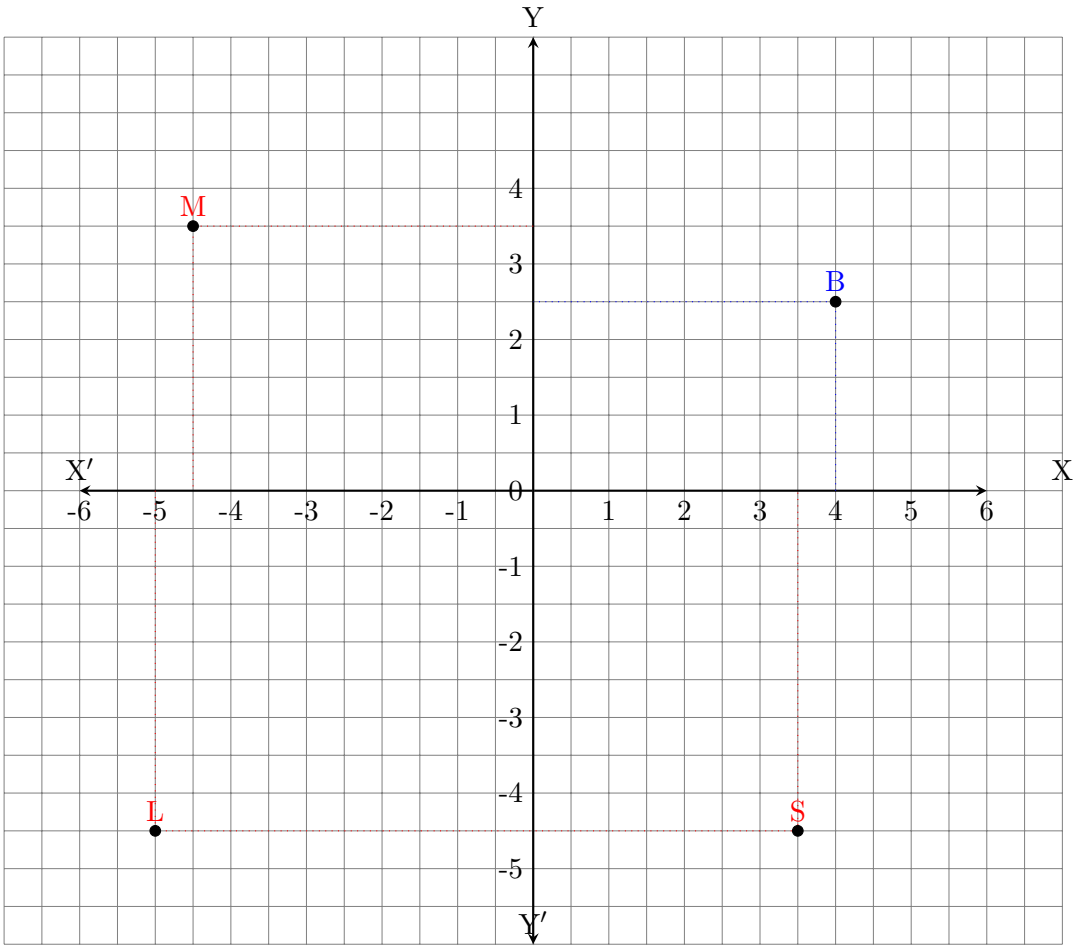



Figure 3.3.5

ii The x-coordinate and y-coordinate of the the point M are $-4.5$ and $3.5$ respectively. Hence, the coordinate of the point M are $(-4.5, 3.5)$.

iii The x-coordinate and y-coordinate of the the point L are $-5$ and $-4.5$ respectively. Hence, the coordinate of the point L are $(-5, -4.5)$.

iv The x-coordinate and y-coordinate of the point S are 3 and $-4.5$ respectively. Hence, the coordinate of the point S are $(3, -4.5)$.

**Example 3.3.2.** Write the coordinate of the points marked on the axes in the Fig.(3.3.6),

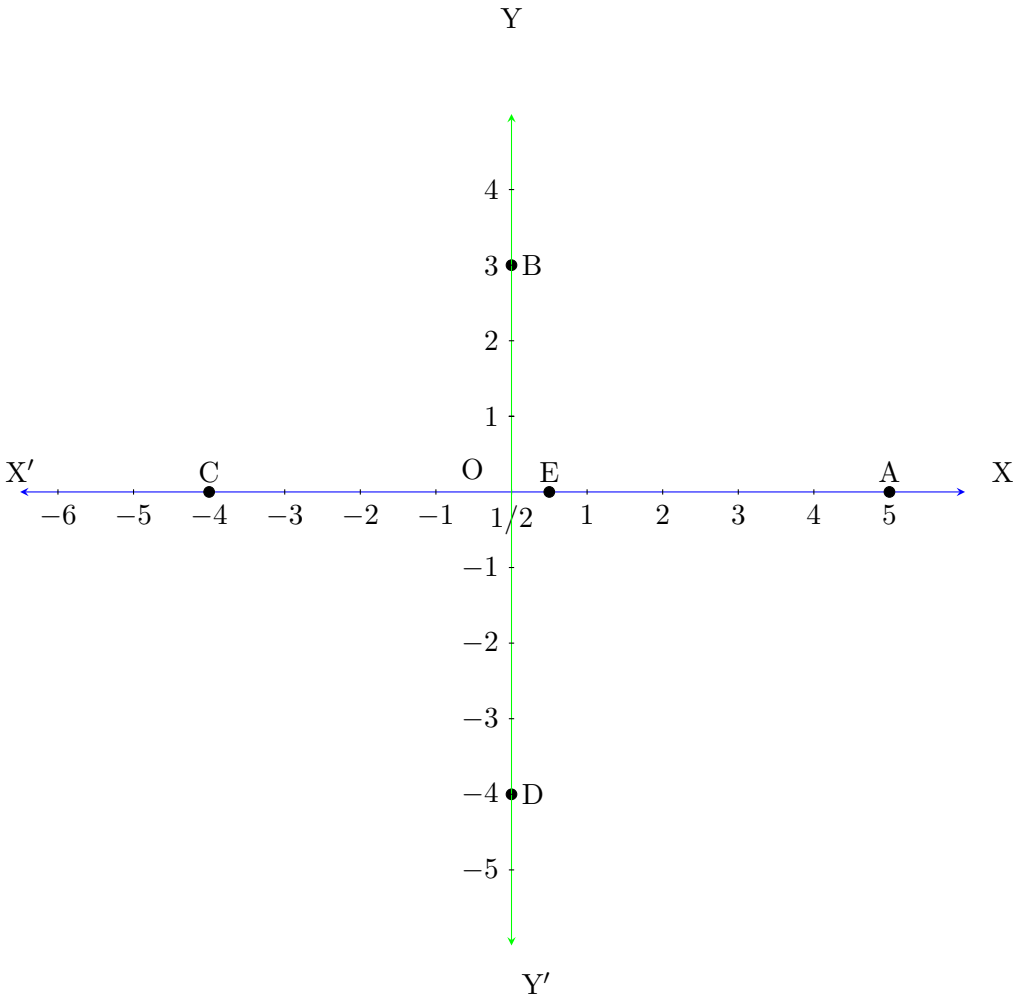



Figure 3.3.6

**Solution:** From Fig.(3.3.6),

i The distance of the point A from y-axis is +5 and distance from x-axis is 0. Hence the x-coordinate of A is 5 and the y-coordinate is 0. So, the coordinate of A are $(5, 0)$. Similar manner the coordinates of following points are given,

ii The coordinate of B are $(0, 3)$.

iii The coordinate of C are $(-4, 0)$.

iv The coordinate of D are $(0, -4)$.

v The coordinate of E are $(.5, 0)$.

**Remark:** The coordinates of origin are $(0, 0)$ because the distance of origin from x-axis and y-axis are 0.

**Observation:**

i If a point is in the $1^{st}$ quadrant, then the coordinates of the point will be in the form $(+, +)$, since the $1^{st}$ quadrant is enclosed by the positive x-axis and positive y-axis.

ii If a point is in the $2^{nd}$ quadrant, then the coordinates of the point will be in the form $(-, +)$, since the $2^{nd}$ quadrant is enclosed by the negative x-axis and positive y-axis.

iii If a point is in the $3^{rd}$ quadrant, then the coordinates of the point will be in the form $(-,-)$, since the $3^{rd}$ quadrant is enclosed by the negative x-axis and negative y-axis.

iv If a point is in the $4^{th}$ quadrant, then the coordinates of the point will be in the form $(+,-)$, since the $4^{th}$ quadrant is enclosed by the positive x-axis and negative y-axis.

## 3.4 Solution of Exercise 3.2

**Problem 3.4.1.** Write the answer of each of the following questions:

i What is the name of horizontal and the vertical lines drawn to determine the position of any point in the Cartesian plane ?

ii What is the name of each part of the plane formed by these two lines ?

iii Write the name of the point where these two lines intersect ?

**Solution:**

i The horizontal line is called x-axis and vertical line is called y-axis which drawn to determine the position of any point in the cartesien plane.

ii x-axis and y-axis are divide the plane in four patrt, which is called I-quadrant, II-quadrant, III-quadrant, IV-quadrant.

iii The intersection of x-axis and y-axis is called origin.

**Problem 3.4.2.** See the Fig.(3.3.7), and write the following:

i The coordinate of B.

ii The coordinate of C.

iii The point identified by the coordinate $(-3, -5)$.

iv The point identified by the coordinate $(2, 4)$.

v The coordinate of D.

vi The coordinate of H.

vii The coordinate of L.

viii The coordinate of M.

**Solution:**

i The distance of point B from x-axis is 2 unit in positive direction and y-axis is 5 unit in negative direction. Hence the coordinates of the point B are $(-5, 2)$.

ii The distance of point C from x-axis is 5 unit in negative direction and y-axis is 5 unit negative direction. Hence the coordinates of the point B are $(5, -5)$.

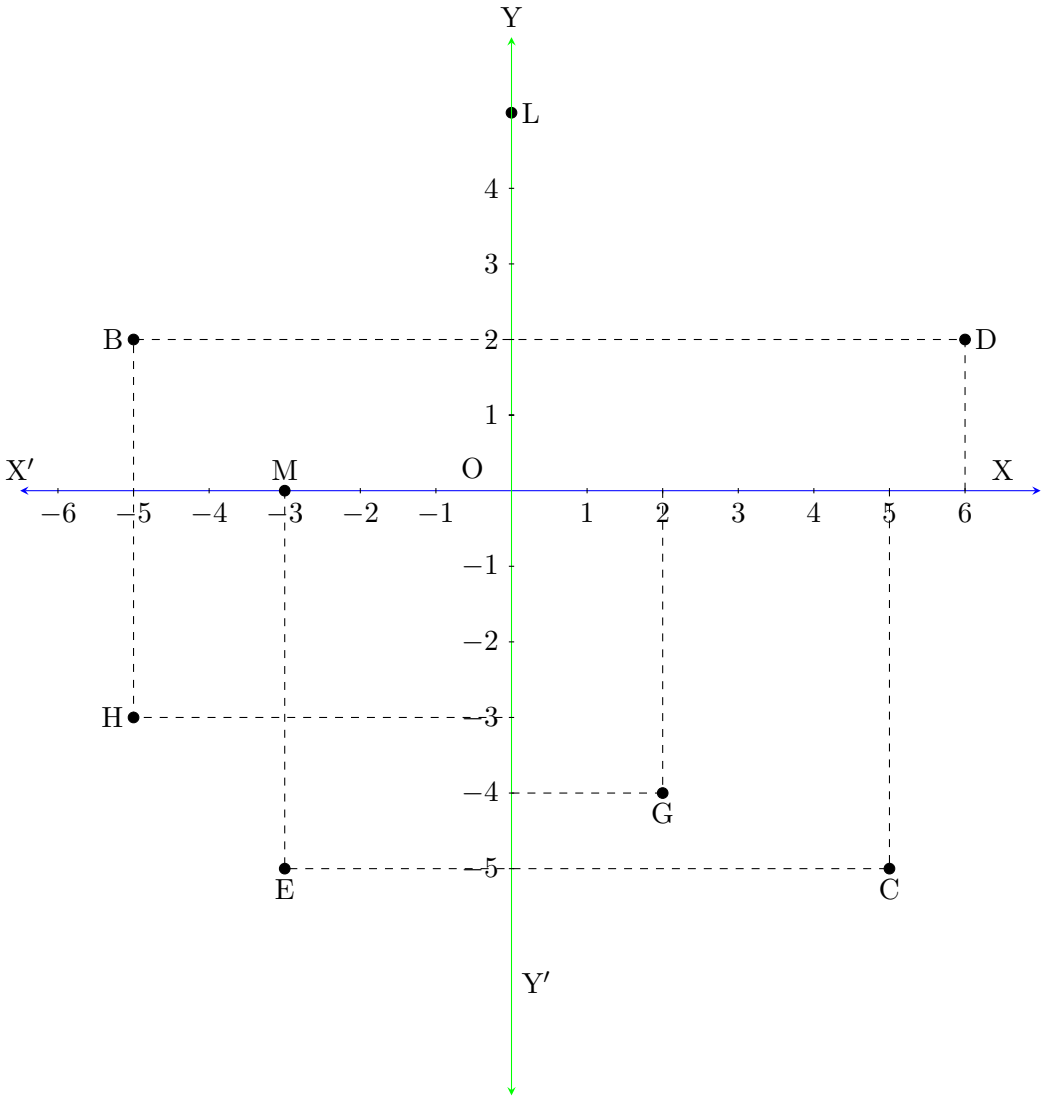



Figure 3.4.1

iii The distance of the point E have 5 unit in negative direction from x-axis and 3 unit in negative direction from y-axis. Hence, the point identified by the coordinates $(-3, -5)$ is E.

iv The distance of the point E has 4 unit in negative direction from x-axis and 2 unit in positive direction from y-axis. Hence, the point identified by the coordinates $(2, -4)$ is G.

v The distance of point D from x-axis is 2 unit in positive direction and y-axis is 6 unit in negative direction. Hence the coordinates of the point B are $(6, 2)$.

vi The distance of point H from x-axis is 3 unit in negative direction and y-axis is 5 unit negative direction. Hence the coordinates of the point B are $(-5, -3)$.

vii The distance of point L from x-axis is 5 unit in positive direction and y-axis is 0 unit. Hence the coordinates of the point B are $(0, 5)$.

viii The distance of point B from x-axis is 0 unit and y-axis is 3 unit in negative direction. Hence the coordinates of the point B are $(-3, 0)$.

# 4 LINEAR EQUATIONS IN TWO VARIABLES

## 4.1 Introduction

We have familiar in earlier classes with linear equations in one variable. They are in the form $x + 2 = 0$, $x - \sqrt{5} = 0$, $\pi x + 7 = 0$, $x = 0$. Also we have seen that these equations have unique solutin in real number system. Now there are questions as what is the form of linear equations in two variables, the solution of these equations will be unique or not. Answer of all such type questions will be given in this chapter.The visualization of equatios and its solutions will be consider in this chapter also.

## 4.2 Linear Equations

Consider the following equations,

$$3x + 2 = 0$$

and

$$2s + 7 = 0$$

the solutions of these two equations are a$x = -\frac{2}{3}$, $x = -\frac{7}{2}$. It can represents on number line uniquely. Some properties of equation; the Solution of equation is not affected when:

i the same number is added to (or subtracted from) bothe sides of the equation.

ii you multiply or divide both the side of the equationby the same non-zero number.

The equation in form $ax + by + c = 0$ is standared form of **linear equation in two variables** where $a, b$ and $c$ are real numbers, and $a$ and $b$ are not both zero.

**Example 4.2.1.** Write each of the following equations in the form $ax+by+c = 0$ and indicate the values of $a, b$ and $c$in each case:

**Solution:** In the frist column given equation and second column the equation written in the standed form $ax + by + c = 0$.

| *equation* | *standard form* | *values of* $a, b, c$ |
|---|---|---|
| $(i)\ 2x + 3y = 4.37$ | $2x + 3y - 4.37 = 0$ | $a = 2, b = 3, c = -4.37$ |
| $(ii)\ x - 4 = \sqrt{3}y$ | $x - \sqrt{3}y - 4 = 0$ | $a = 1, b = \sqrt{3}, c = -4$ |
| $(iii)\ 4 = 5x - 3y$ | $5x - 3y - 4 = 0$ | $a = 5, b = -3, c = -4$ |
| $(iv)\ 2x = y$ | $2x - y = 0$ | $a = 2, b = -1, c = 0$ |

**Example 4.2.2.** Write each of the following as an equation in two variables:

i $x = -5$

ii $y = 2$

iii $2x = 3$

iv $5y = 2$

**Solution:** The standarded form of equation given in tabular form.

| *equation* | *standard form* |
|---|---|
| $(i)\ x = -5$ | $x + 0y + 5 = 0$ |
| $(ii)\ y = 2$ | $0x + y - 2 = 0$ |
| $(iii)\ 2x = 3$ | $2x + 0y - 3 = 0$ |
| $(iv)\ 5y = 2$ | $0x + 5y - 2 = 0$ |

## 4.3 Solution of Exercise 4.1

**Problem 4.3.1.** The cost of a notebook is twice the cost of a pen. Write a linear equation in two variable to represent this statement.

**Solution:** Let the cost of a notebook is Rs.$x$ and cost of a pen is Rs. $y$. Then according to the question,

$$x = 2y.$$

**Problem 4.3.2.** Express the following linear equations in the form $ax + by + c = 0$ and indicate the values of $a, b$ and $c$ in each case:

i $2x + 3y = 9.3\bar{5}$

ii $x - \frac{y}{5} - 10 = 0$

iii $-2x + 3y = 6$

iv $x = 3y$

v $2x = -5y$

vi $3x + 2 = 0$

vii $y - 2 = 0$

viii $5 = 2x$

**Solution:** The standard form of given equations and there coefficients are:

| *Equation* | *Standard form* | *Values of* $a, b, c$ |
|---|---|---|
| $(i)\ 2x + 3y = 9.3\bar{5}$ | $2x + 3y = 9.3\bar{5}$ | $a = 2, b = 3, c = 9.3\bar{5}$ |
| $(ii)\ x - \frac{y}{5} - 10 = 0$ | $x - \frac{y}{5} - 10 = 0$ | $a = 1, b = -\frac{1}{5}, c = -10$ |
| $(iii)\ -2x + 3y = 6$ | $-2x + 3y - 6 = 0$ | $a = -2, b = 3, c = -6$ |
| $(iv)\ x = 3y$ | $x - 3y + 0 = 0$ | $a = 1, b = -3, c = 0$ |
| $(v)\ 2x = -5y$ | $2x + 5y + 0 = 0$ | $a = 2, b = 5, c = 0$ |
| $(vi)\ 3x + 2 = 0$ | $3x + 0y + 2 = 0$ | $a = 3, b = 0, c = 2$ |
| $(vii)\ y - 2 = 0$ | $0x + y - 2 = 0$ | $a = 0, b = 1, c = -2$ |
| $(viii)\ 5 = 2x$ | $2x + 0y - 5 = 0$ | $a = 2, b = 0, c = -5$ |

## 4.4 Solution of a Linear Equation

We have been seen that every linear equation in one variable has unique solution. But what we can say about the solution of linear equation in two variables. Frist there is a question that what are the solution of linear equation in two variable. The values of $x$ and $y$ which satisfy the linear equation in two variable is called the **solution** of equation.

**Remark:**

- There are infinite number of solutions to a linear equation in two variables.

**Note:** The notation $(.,.)$ means frist coordinate is value of $x$ and second coordinate is value of $y$.

**Example 4.4.1.** Find four solution of the equation $x + 2y = 6$.

**Solution:** Now by inspection, take $x = 4, y = 1$, satisfy the equation, so $(4, 1)$ is a solution of the equation $x + 2y = 6$. As,

$$x + 2y = 4 + 2 \times 1 = 6$$

Consider $x = 2$ then $2 + 2y = 6 \implies 2y = 4 \implies y = 2$. Hence $(4, 2)$ is a solution. Similar manner if $x = 6$ then $6 + 2y = 6 \implies 2y = 0 \implies y = 0$. Hence $(6, 0)$ is solution. If $x = 0$ then $0 + 2y = 6 \implies y = \frac{6}{2} \implies y =$. Hence

$(0, 3)$ is the solution of equation.

**Example 4.4.2.** Finf two solutions for each of the following equations:

i $4x + 3y = 12$

ii $2x + 5y = 0$

iii $3y + 4 = 0$

**Solution:**

i Taking $x = 0$, We get $4 \times 0 + 3y = 12 \implies 3y = 12 \implies y = 4$, so $(0, 4)$ is a solution of equation. If we take $y = 0$, we get $x = 3$ so $(4, 0)$ second solution of equation.

ii Taking $x = 0$, we get $5y = 0 \implies y = 0$. Hence $(0, 0)$ is a solution of given equation. Again take $x = 1$ we get $2 + 5y = 0 \implies y = -\frac{2}{5}$. Hence $\left(1, -\frac{2}{5}\right)$ is a solution.

iii The given equation is $3y + 4 = 0$. In this equation $x$ variable is not present hence $x$ can take any values. From the equation we get $y = -\frac{4}{3}$, therefore $(0, -\frac{4}{3}), (1, -\frac{4}{3})$ are two solution of given equation.

## 4.5 Solution of Exercise 4.2

**Problem 4.5.1.** Which one of the following option is true, and why?

$y = 3x + 5$ has

i a unique solution,

ii only two solution,

iii infinitely many solutions.

**Solution:** The equation $y = 3x + 5$ has infinite many solutions so option (iii) is true. Because for each $x$, equation gives a value of $y$.

**Problem 4.5.2.** Write four solutions for each of the following equations:

i $2x + y = 7$

ii $\pi x + y = 9$

iii $x = 4y$

**solution:**

i Take $x = 0$, we get $y = -2x + 7 \implies y = -2 \times 0 + 7 \implies y = 7$. Hence $(0, 7)$ is a solution. Next take $y = 0$, so $2x + 0 = 7 \implies x = \frac{7}{2}$. Hence $(\frac{7}{2}, 0)$ is another solution. Now take $x = 1$, then we get $2 \times 1 + y =$

$7 \implies y = 5$ so $(1, 5)$ is a solution also. Next take $x = -1$, then $2(-1) + y = 7 \implies y = 9$. Hence $(-1, 9)$ is a solution.

ii Take $x = 0$, we get $y = -\pi x + 9 \implies y = -\pi \times 0 + 9 \implies y = 9$. Hence $(0, 9)$ is a solution. Next take $y = 0$, so $\pi x + 0 = 9 \implies x = \frac{9}{\pi}$. Hence $(\frac{9}{\pi}, 0)$ is another solution. Now take $x = 1$, then we get $\pi \times 1 + y = 9 \implies y = 9 - \pi$ so $(1, 9 - \pi)$ is a solution also. Next take $x = -1$, then $\pi(-1) + y = 9 \implies y = 9 + \pi$ Hence $(-1, 9 + \pi)$ is a solution.

iii Take $x = 0$, we get $4y = x \implies y = 0$. Hence $(0, 0)$ is a solution. Next take $y = 1$, so $4 \times 1 = x \implies x = 4$. Hence $(4, 1)$ is another solution. Now take $y = 2$, then we get $x = 4 \times 2 \implies x = 8$ so $(8, 2)$ is a solution also. Next take $y = -1$, then $x = (-1)4 \implies x = -8$. Hence $(-8, -1)$ is a solution.

**Problem 4.5.3.** Check which of the following are solutions of the equation $x - 2y = 4$ and which are not:

i $(0, 2)$

ii $(2, 0)$

iii $(4, 0)$

iv $(\sqrt{2}, 4\sqrt{2})$

v $(1, 1)$

**Solution:**

i now putting $x = 0, y = 2$ in the equation $x - 2y = 4$. We get $0 - 2 \times 2 = 4 \implies -4 = 4$, which absurd. So $(0, 2)$ is not solution of the equation $x - 2y = 4$.

ii now putting $x = 2, y = 0$ in the equation $x - 2y = 4$. We get $2 - 2 \times 0 = 4 \implies 2 = 4$, which absurd. So $(2, 0)$ is not solution of the equation $x - 2y = 4$.

iii now putting $x = 4, y =$ in the equation $x - 2y = 4$. We get $4 - 2 \times 0 = 4 \implies 4 = 4$, So $(0, 2)$ is a solution of the equation $x - 2y = 4$.

iv now putting $x = \sqrt{2}, y = 4\sqrt{2}$ in the equation $x - 2y = 4$. We get $\sqrt{2} - 2 \times 4\sqrt{2} = 4 \implies -7\sqrt{2} = 4$, which absurd. So $(0, 2)$ is not solution of the equation $x - 2y = 4$.

v now putting $x = 1, y = 1$ in the equation $x - 2y = 4$. We get $1 - 2 \times 1 = 4 \implies -1 = 4$, which absurd. So $(1, 1)$ is not solution of the equation $x - 2y = 4$.

**Problem 4.5.4.** Find the value of $k$, if $x = 2, y = 1$ is a solution of the equation $2x + 3y = k$.

**Solution** If $x = 2, y = 1$ is a solution of the equation $2x + 3y = k$. Then value of $x, y$ will satisfy the equation. Now puttinf the value of $x, y$ in $2x + 3y = k$. We get $2x + 3y = k \implies 2 \times 2 + 3 \times 1 = k \implies k = 4 + 3 = 7$. The value of $k = 7$.

# 5 INTRODUCTION TO EUCLID'S GEOMETRY

## 5.1 Introduction:

In the mathematics world, 'geometry' word give the image of their development and place of birth. Geometry made from two word 'geo' and 'metriein' which are Greek word, it reffer as measure of earth. So we arrived a conclucion that this branch of mathematics development starts in Egipt. Later other part of world gave their contribution to make rich as a displine. The development foot print of geometry in babylonia, china, india apears.

The first proof in geometry given by Greek mathematician Thales as "a circle is bisected by its diameter.Thales's pupil Pythagoras (572 BCE) and his group discovered many geometric properties and developed the theory of geometry. Next mathematician Euclid which was teacher at Alexandria in Egypt organised all the theorems and properties till 300 BCE in his book 'Elements'.

In this chapter, we will study Euclid's approach to geometry and shall try to link it with the present day geometry.

## 5.2 Euclid's Definitions, Axioms and Postulates

At the time Greek mathematician Euclid study of geometry in abstract form mostly. They consider the solid as solid has shape, size, position and can be moved from one placed to another. Its boundaries are called **surfaces**. They separate one part of the space from another, are said to have no thickness. The boundaries of the surfaces are **curves** or **lines**. These lines end in **points**.

Now in real world any solid object has three extentions which is called a **dimension.** Hence a solid has three dimensions, a surface has two dimension, a line has one dimension and a point has none dimension. Euclid organise these statments as definitions in his book **Elements**. A few of them are given below:

i A **point** is that which has no part.

ii A **line** is breadthless length.

iii The end of a line are **points**.

iv A **straight line** is a line which lies evenly with the points on itself.

v A **surface** is that which has length and breadth only.

vi The edge of a surface are **lines**.

vii A **plane surface** is a surface which lies evenly with the straight line on itself.

If you study carefully these definitions there are many terms used, which need to define them. Next you need to define terms involve in definitions of terms. So this sequence of definition will continue. Hence some geometric terms consider **undefined**. So in geometry, you should take *a point, a line and a plane (in Euclid word a plane surface) as undefined terms.* But these terms can be explain them with help of ***'physical models'***.

To develop the geometry and define definition Euclid assumed certain properties, which were not to be proved. But these assumptions are actually *"obvious universal truths"*. He divided them into two types: axioms and postulates. To Geometrical assumption he called **'postulate'** and common notions that not spacialy related to only geometry and use throughout mathematics is called **'axioms'**.

There are some Euclid's Axioms:

i Things which are equal to same thing are equal to another.

ii If equals are added to equals, the whole are equal.

iii If equals are subtracted from equal, the remainder are equal.

iv Things which coincide with one another are equalto one another.

v The whole is greater than the part.

vi Things which are double of same things are equal to one another.

vii Things which halves of the same things are equal to one another.

### Application of Axiom

The concept of application of axiom:

i The Seven Axiom are applicable for 'common notion' to magnitudes of some kind. The Frist common notion could be applied to plane figures as area of plane figure.

ii The magnitude of the same kind can be compared and added but magnitude of different kinds can not be compared. For example, a line cannot be compared to a rectangle, nor can angle be compared to pentagon.

iii The $4^{th}$ axiom say that if two things are identical, then they are equal. It can be say that, everything equal itself.

iv Axiom $5^{th}$ give concept of 'greater than'. It means if quantity B is part of quantity A, Then $A > B$

**Postulate 1 :** *A straight line may be drawn from any one point to any other point.*

This postulate tells us that there is a *unique line joining two distinct points.*

**Axiom 5.1 :** *Given two distinct points, there is a unique line that passes through them.*

Figure 5.2.1: Unique line through two points

In the Fig.(5.2.1) there are only one line passing through P also pass through Q that is PQ. next there are only one line passing through Q also pass through P that is PQ. Hence Axiom 5.1 is self-evident.

**Postulate 2 :** *A terminated line can be produced indefinitely.*

This postulate say that any line segment can be extended on either side to form a line.

**Postulate 3 :** *A circle can be drawn with any centre and any radius.*

Figure 5.2.2: Extention of line segment

**Postulate 4 :** *All right angle are equal to one another.*

**Postulate 5 :** *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced infinitly, meet on that side on which the sum of angle is less than two right angles.*

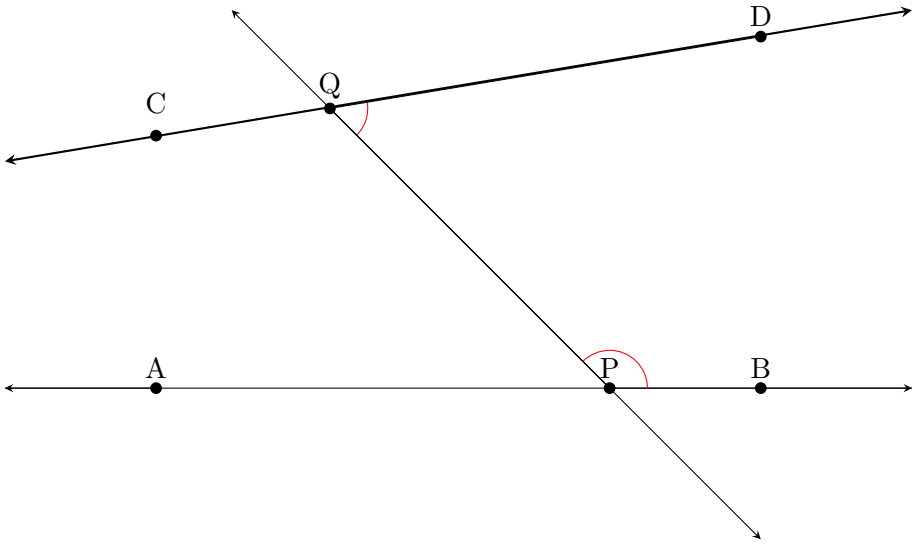



Figure 5.2.3: Condition of intersecting lines

For example line PQ in the Fig.(5.2.3) falls on line AB and CD such that the interior angle at Q and P is greater than $180^0$ on the right of PQ. Therefore, the lines AB and CD will not intersect on left right side of PQ.

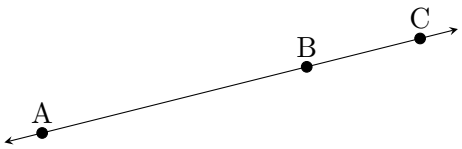



Figure 5.2.4: Colinearty condition of three point

On the basis on his postulate and axiom Euclid prooved other results. Then, the basis on these results he proved more results applying deductive reasoning. These proved statements are called **propositions or theorems**.

**Example 5.2.1.** If A, B and C are three points on a line, and B lies between A and C (With Fig.5.2.4), then prove that $AB + BC = AC$.

**Solution:** In the Fig.(5.2.4) AC coincide with AB+BC. Hence by Euclid Axiom (4), tells that things which coincide with one another are equal to one another. So, it can be deducted that

$$AB + BC = AC.$$

**Example 5.2.2.** Prove that an equilateral tringle can be constructed on any given line segment.

**Solution:** In the given statement, a line segment of any length is given, Say AB, in Fig.(5.2.5).

Here we construct a line segment AB. From Euclid's postulate 3, we can draw a circle with radius AB and at center A. Similarly, we draw another ciecle with radius BA and at center B. These two circles meet at a points, say

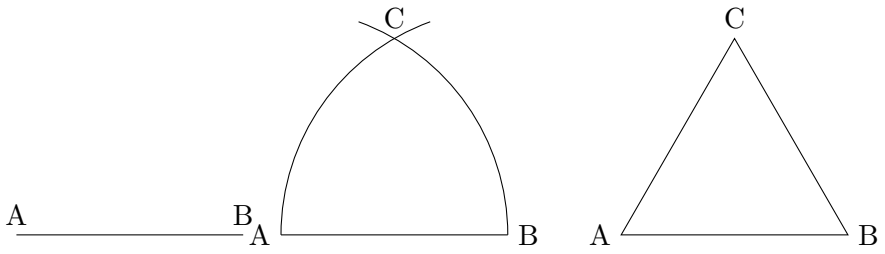



Figure 5.2.5: Existence of equilateral triangle

C. Next drawing the line segment AC and BC to form $\triangle$ ABC.

Now we prove that this triangle is equilateral triangle, i.e, AB=AC=BC.

(1) Here, AB=AC, since they are the radii of same circle.

(2) Similarly, AB=BC, raddi of same circle.

From these two fact and Euclid's axiom (3) that things which are equal to same thing are equal to another, we conclude that AB=BC=AC. So $\triangle$ ABC is an equilateral triangle.

**Theorem 5.2.3:** *Two distinct lines connot have more than one point in common.*

*Proof.* At the same time, let us suppose that the two lines intersec in two distinct points, say P and Q. So, we find two lines passing thruogh two distinct points P and Q. But our assumption clashes with the axiom (5.1) that only one line can pass through two distinct points. Hence, the assumption that we started with, that two lines can pass through through two distinct points is worng. From this, we arrived conclusion that two distinct lines cannot have

more than one point in common. □

## 5.3 Solution of exercise 5.1

**Problem 5.3.1.** Which of the following statements are true and which are false ? Give reasons for your answer:

i Only one line can pass through a single point.

ii There are an infinite number of lines which pass through two distinct points.

iii A terminated line can be produced indefinitely on both the sides.

iv If two circle are equal, then their radii are equal.

v In Fig.(5.3.1), If AB=PQ and PQ=XY, then AB=XY.

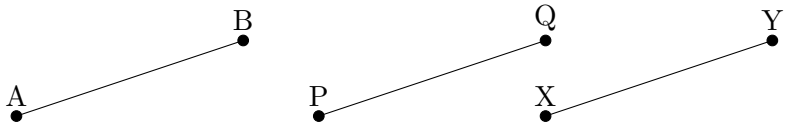



Figure 5.3.1

**Solution:**

i False. The postulate (1) tells that any draw from one point to one another point. So if you fixed a point, then there are infinite number points around its.

ii False. According axiom (5.1) only one line can be draw from two distinct points.

iii True. According postulate (2) a terminated line can be extended on either side to form a line.

iv True. According axiom (4) if two things are identical, then they are equal. So their radii will be equal.

v True. In the given Fig.(5.3.1), given that AB=PQ and PQ =XY. So from Axiom (1), AB=XY.

**Problem 5.3.2.** Give a definition for each of the following terms. Are there other terms that needed to be define first ? What are they and how might you define them ?

i Parallel lines.

ii Perpendicular lines.

iii Line segment.

iv Radius of a circle.

v Square.

**Solution:** To explain the following terms we need define a quantity 'angle'. The angle between to lines is difference of slope these two lines from given a fixed line.

i Two lines are called parallel line if angle between them is zero degree.

ii Two lines are called perpendicular if angle between them is $90^0$ or a right angle.

iii A line segment is a part of line which two terminating ends are points.

iv The distance from center to any point on circumference of the circle is called radius of circle.

v An quadrilateral whose each side have same length and each angles are rightangle.

**Problem 5.3.3.** Consider two 'postulates' given below:

i Given any two distinct points A and B, there exists a third point C which is in between A and B.

ii There exist at least three points that are not on the same line.

Do these postulates contain any undefined terms ? Are these postulates consistent ? Do they follow with Euclid'postulates ? Explain.

**Solution:** Yes, these postulates contain one undefine terms which is 'distinct point'. Yes, these postulates consistent. Yes, they follow from Eucid's postulates.

**Explanation:** How to define distinct point because according to Euclid's definition, a point has no part. These postulates does not clashes any postulates given by Euclid. These postulate contained in the Axiom (5.1).

**Problem 5.3.4.** If a point C lies between two points A and B such that AC=BC, then prove that $AC = \frac{1}{2}AB$. Explain by drawing the figure.

*Proof.* In the Fig.(5.3.2), given that AC=BC. So, 2AC=2BC from axiom (vi). Now AB= 2AC, and AB = 2BC, so AB =2AC=2BC next $\frac{1}{2}AB = AC$ and $\frac{1}{2}AB = BC$ from axiom(vii) $\frac{1}{2}AB = AC$.

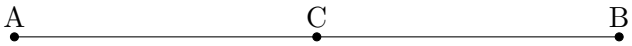



Figure 5.3.2

□

**Problem 5.3.5.** In Problem (5.3.4) point C is called a mid point of line segment AB. Prove that every line segment has one and only one mid-point.

*Proof.* In Fig.(5.3.3), suppose two point C and D are mid-point of line segment AB. From result problem (5.3.4) we got $\frac{1}{2}AB = AC$ and $\frac{1}{2}AB = AD$ because C and D bothe are mid-point of line AB. So Euclid's axiom (i), AD=AC. Hence D and C are identical points. After considaration of these fact we arrived the conclusion that every line segment has one and only one mid-point. □

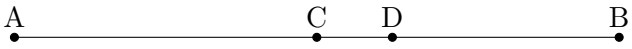



Figure 5.3.3

**Problem 5.3.6.** In Fig.(5.3.4), if AC=BD, then prove that AB=CD.

Figure 5.3.4

*Proof.* In Fig.(5.3.4), given that AC = BD. Next we subtract BC to bothe side in equation AC = BD. So AC-BC=BD-BC. Hence with axiom (iii), AB=CD. □

**Problem 5.3.7.** Why is axiom(v), in the list of Euclid's axioms, considered a 'universal truth' ? (Note that the question is not about the fifth postulate.)

**Solution:** The axiom (v), is true for any quantity of universe, so it is called 'universal truth'.

# 6 LINES AND ANGLES

## 6.1 Introduction

In the previous chapter 5, you have learned about points, and lines. You have also learned some axioms and you proved many statements with help of axioms. In this chapter you will study the properties of the angles formed when two lines intersect each other, and also the properties of the angles when two or more parallel lines intersected by a line at two distinct points. Using deductive reasoning you will prove some statements using these propertis.

This knowlwdge needs in your daly life uses and sciensce study. In next chapter of geometry, you will need these propertis of lines and angles to deduce more and more useful properties and statements.

## 6.2 Basic Terms and Definitions and Notations

**Line-segment and Rays:** A part (or portion) of a line with two end points is called a **line-segment** and a part of a line with one end point is called a

**ray**.

*Remark* 6.2.1. The line-segment AB is denoted by $A\bar{B}$, and its length is denoted by AB. The ray denoted by $\vec{AB}$, and a line denoted by $\overleftrightarrow{AB}$. However, we will not use these symbols, and will denote the line segment AB, ray AB, length AB and line AN by same symbol, AB.

**Collinear points and Non-collinear points:** If three or more points lie on the same line, they are called **collinear points**, otherwise they are called **non-collinear points**.

**Angles, Arms and Vertex:** When two rays orginated from same end point, they formed **angle**. These rays which forms angle is called **arms** of angle and the end point is called **vertex** of the angle.

**Types of Angles:** An **acute angle** measures between $0^\circ$ and $180^\circ$. A **right angle** exactely equal to $90^\circ$.

An angle graeter than $90^\circ$ but less than $180^\circ$ is called an **obtuse angle**. A **straight angle** equal to $180^\circ$.

An angle which is greater than $180^\circ$ but less than $360^\circ$ is called a **reflex angle**.

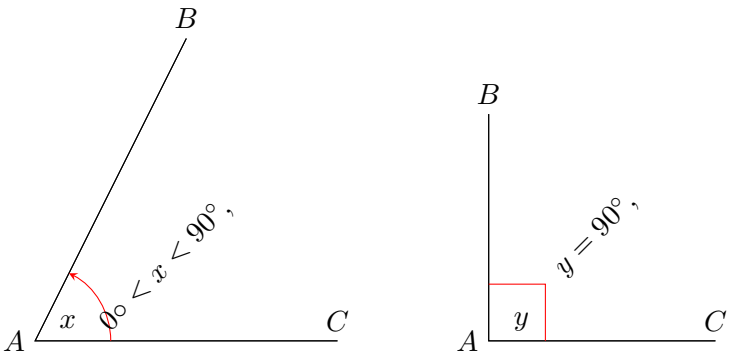



Figure 6.2.1: Acute angle and Right angle

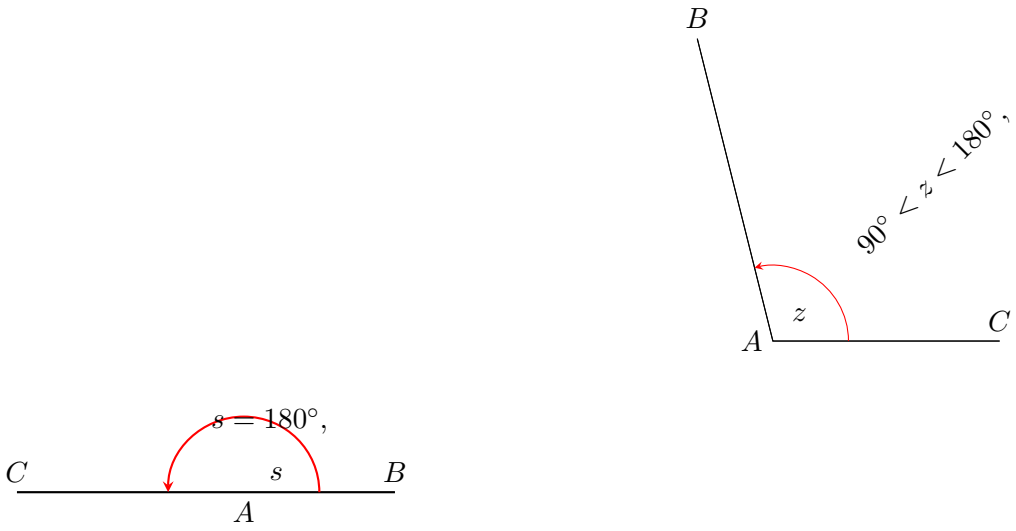



Figure 6.2.2

Two angles are **adjacent**, if they have a common vertex, a common arm and their non-common arms are on different sides of the common arm (See Fig.6.2.2).

The two angle whose sum is 90° are called **complementry angles**, two angle whose sum is 180° are called **supplementary angles**.

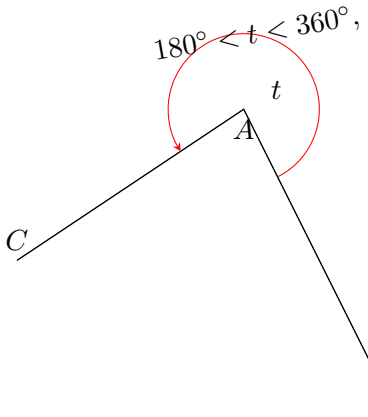



Figure 6.2.3

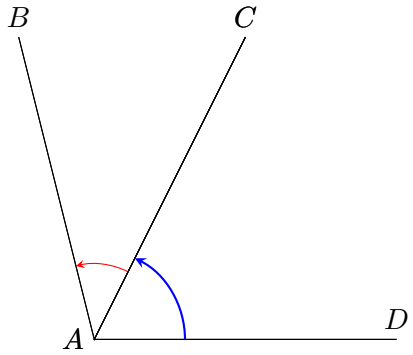



Figure 6.2.4: Adjacent angles

**Vertically opposite angle:** When two line intersect each other they form vertically opposite angle.

**Axiom 6.1:** If a ray stands on a line, then the sum of two adjacent angle so formed is 180°.

**Note :** If the sum of two adjacent angle 180°, then a ray stands on line.

**Axiom 6.2 :** If the sum of two adjacent angle is 180°, then the non-

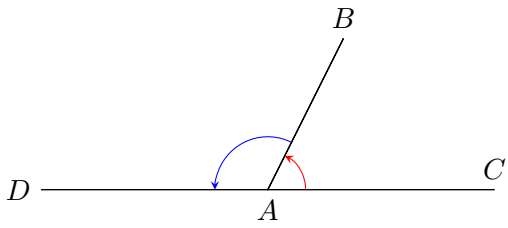



Figure 6.2.5: Linear pair angles

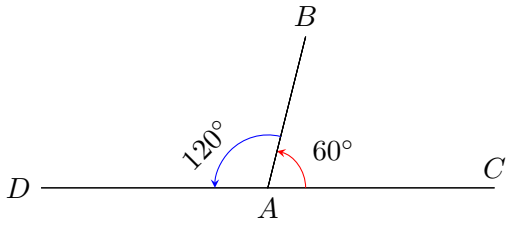



Figure 6.2.6: Supplementary angles

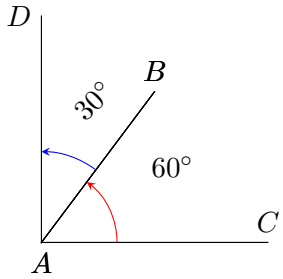



Figure 6.2.7: Complentary angles

common arms of angles form a line.

The axiom (6.1) and (6.1) together is called the **Linear Pair Axiom**.

**Theorem 6.2.2:** *If two line intersect each other, then the vertically opposite angles are equal.*

*Proof.* In the theorem (6.2.2) it is given that 'two lines intersect each other'.

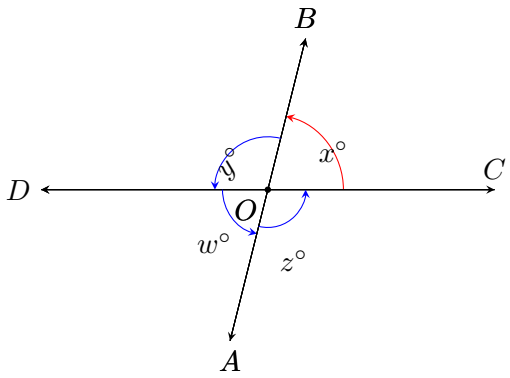



Figure 6.2.8: Vertically opposite angles

So, let PQ and RS be two lines intersecting at O as given in Fig.(6.2.9).

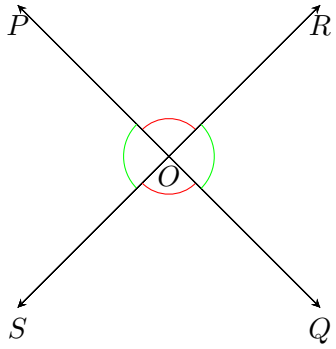



Figure 6.2.9: Vertically opposite angles

They lead to two pairs of vertically opposite angles, namely,

(i) $\angle POS$ and $\angle ROQ$ (ii) $\angle ROP$ and $\angle SOQ$.

We need to prove that $\angle POS = \angle ROQ$ and $\angle ROP = \angle SOQ$.

Now ray OP stands on line RS. Therefore,

(6.2.1) $$\angle POS + \angle ROP = 180^\circ$$

(Linear pais axiom)

$$\angle SOR + \angle QOR = 180^\circ \tag{6.2.2}$$

From equation (6.2.1) and (6.2.2) we get

$$\angle POS + \angle ROP = \angle ROP + \angle QOR$$

Thisimplies $\angle POS = \angle QOR$

Similarly, it can be proved that $\angle POR = \angle QOS$. □

**Example 6.2.3.** In the Fig. (6.2.10) Line PQ and RS intersect each other at point O. If $\angle POR : \angle ROQ = 5 : 7$, find all the angles.

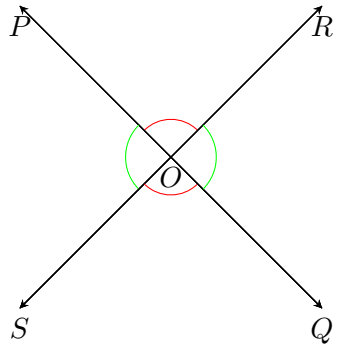



Figure 6.2.10

**Solution :** From theorem (6.2.2), we have $\angle POR : \angle ROQ = 180^\circ$ as Linear pair of angle.

But given that $\angle POR : \angle ROQ = 5 : 7$.

Therefore,

$$\angle POR = \frac{5}{12} \times 180^\circ = 75^\circ$$

Similarly,

$$\angle ROQ = \frac{7}{12} \times 180^\circ = 105^\circ$$

Now, $\angle POR = \angle SOQ = 75^\circ$ because they are vertically opposite angle. $\angle ROQ = \angle POS = 105^\circ$ because they are vertically opposite angle.

**Example 6.2.4.** In Fig. (6.2.11) ray OS stands on a line POQ. Ray OR and ray OT atr angle bisectors of $\angle POS$ and $\angle SOQ$, respectively. If $\angle POS = x$, find $\angle ROT$.

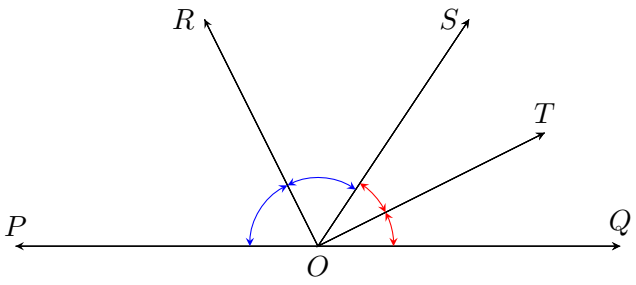



Figure 6.2.11

**Solution :** Ray OS stands on a line POQ. Therefore,

$$\angle SOQ + \angle SOP = 180^\circ$$

Given, $\angle POS = x$. Therefore,

$$x + \angle SOQ = 180^\circ$$

So,

$$\angle SOQ = 180^\circ - x$$

Now, ray OR bisects $\angle POS$, therefore,

$$\angle ROS = \frac{1}{2} \times \angle POS = \frac{1}{2} \times x = \frac{x}{2}$$

Similaraly,

$$\angle SOT = \frac{1}{2} \times \angle SOQ = \frac{1}{2} \times (180 - x) = 90^\circ - \frac{x}{2}$$

Now,

$$\angle ROT = \angle ROS + \angle SOT \implies \angle ROT = \frac{x}{2} + 90^\circ - \frac{x}{2}$$

$$\angle ROT = 90^\circ$$

**Example 6.2.5.** In Fig.(6.2.12) OP, OQ, OR, and OS are four rays. Prove that $\angle POQ + \angle QOR + \angle SOR + \angle POS = 360^\circ$

**Solution :** In Fig.(6.2.12), you need to produced any of the rays OA, OP, OS, OR backwards a point. Let us produce a ray OQ backwards to a point T, so TOQ forms a line. See the Fig.(6.2.13).

Now, rays OS stands on line TOQ. Therefore

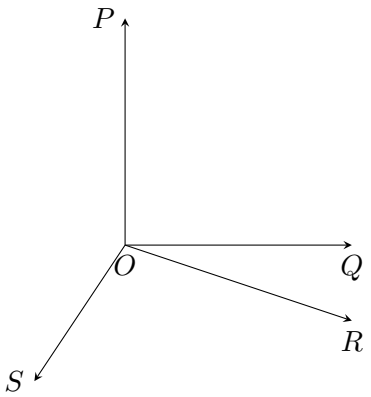



Figure 6.2.12

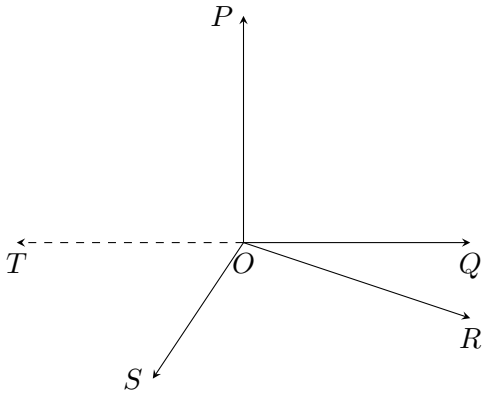



Figure 6.2.13

(6.2.3) $$\angle TOS + \angle SOQ = 180^\circ$$

But $\angle SOQ = \angle SOR + \angle QOR$, so (6.2.3) becomes

(6.2.4) $$\angle TOS + \angle SOR + \angle QOR = 180^\circ$$

Next, rays OS stands on line TOQ. Therefore

$$\angle TOP + \angle QOP = 180^\circ \tag{6.2.5}$$

Now adding equation (6.2.4) and (6.2.5),

$$\angle TOP + \angle QOP + \angle TOS + \angle SOR + \angle QOR = 360^\circ \tag{6.2.6}$$

But, $\angle SOR + \angle QOR = \angle SOQ$. THerefore equation (6.2.6) gives, $\angle TOP + \angle QOP + \angle TOS + \angle SOQ = 360^\circ$

## 6.3 Solution of Exercise 6.1

**Problem 6.3.1.** In Fig.(6.3.1), line XY and MN intersect at O. If $\angle POY = 90^\circ$ and $a : b = 2 : 3$, find $c$.

**Solution :** In Fig(6.3.1), rays OP stands on line XOY and $\angle POY = 90^\circ$. Therefore,

$$\angle POY + \angle POX = 180^\circ$$

But, $\angle POY = 90^\circ$, So

$$\angle POX + 90^\circ = 180^\circ \implies \angle POX = 180^\circ - 90^\circ = 90^\circ$$

Now, $\angle MOX = \frac{3}{5} \times \angle POX \implies \angle MOX = \frac{3}{5} \times 90^\circ = 54^\circ$

The ray OX stands on line MON. So,

$$\angle MOX + \angle XON = 180^\circ \implies c + b = 180^\circ \implies c = 180^\circ - b$$

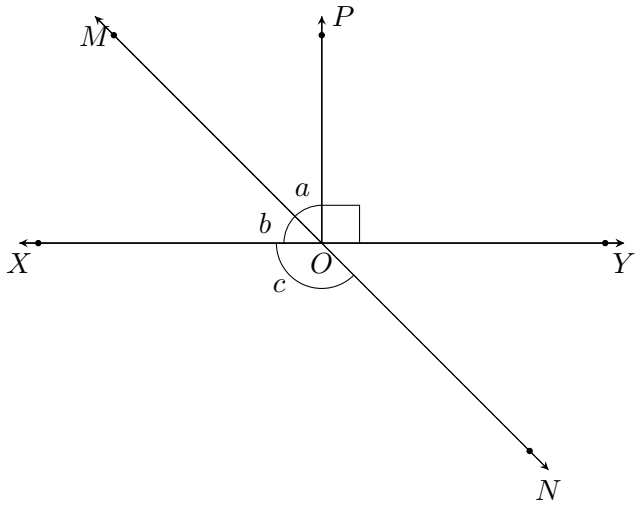



Figure 6.3.1

. Therefore, $c = 180^\circ - 54^\circ = 126^\circ$

**Problem 6.3.2.** In Fig.(6.3.2), lines AB and CD intersect at O. If $\angle AOC + \angle BOE = 70^\circ$ and $\angle BOD = 40^\circ$, find $\angle BOE$ and reflex $\angle COE$.

**Solution :** In Fig.(6.3.2), ray OC stans on the line AOB. Therefore,

$$\angle AOC + \angle COB = 180^\circ$$

$$\angle AOC + \angle EOC + \angle BOE = 180^\circ$$

but, $\angle AOC + \angle BOE = 70^\circ$. So,

$$\angle EOC = 180^\circ - 70^\circ = 110^\circ$$

, Hence reflex $\angle EOC = 360^\circ - 110^\circ = 250^\circ$. The lines AB and CD intersec at the point O. So, $\angle AOC = \angle BOD$.

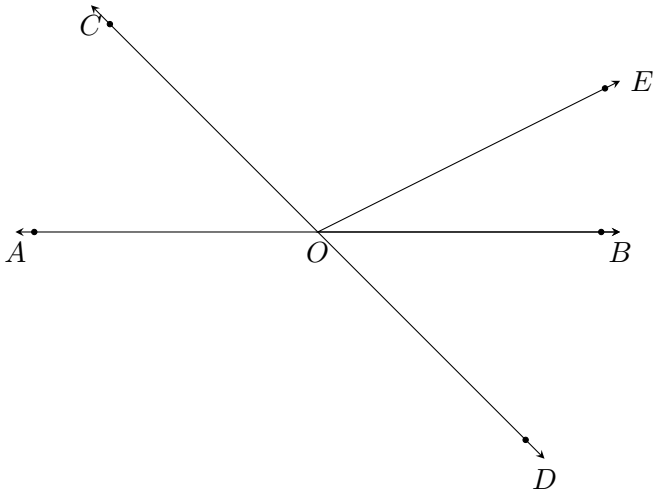



Figure 6.3.2

But given that $\angle BOD = 40°$. Therefore,$\angle AOC = 40°$ Now $\angle BOE = 70° - 40° = 30°$.

**Problem 6.3.3.** In Fig.(6.3.3), $\angle PQR = \angle PRQ$, then prove that $\angle PQS = \angle PRT$.

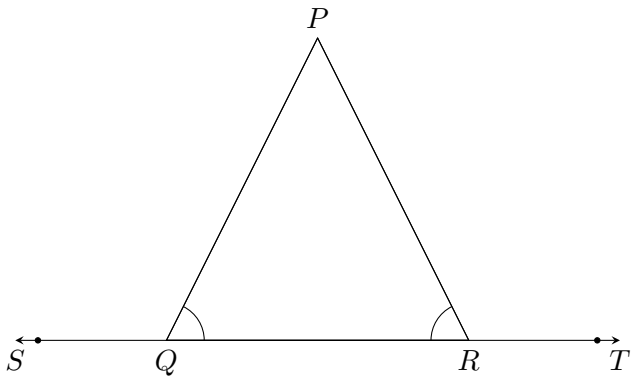



Figure 6.3.3

**Solution :** In Fig.(6.3.3), The line segments QP and RP stand on line SQRT. Therefore, $\angle SQP, \angle PQR$ and $\angle PRQ, \angle PRT$ are linear pair angle. Hence

$$\angle SQP + \angle PQR = 180^\circ$$

$$\angle PRQ + \angle PRT = 180^\circ$$

Therefore,

$$\angle SQP + \angle PQR = \angle PRQ + \angle PRT$$

But, given that, $\angle PQR = \angle PRQ$. So, $\angle PQS = \angle PRT$.

**Problem 6.3.4.** In Fig.(6.3.4), if $x + y = w + z$, then prove that AOB is a line.

**Solution :** In Fig.(6.3.4), rays OA, OB, OC and OD are start from point O. They makes angle $x$, $y$, $z$, $w$ one another. So $x + y + z + w = 360^\circ$, but given that $x + y = w + z$, so

$$2x + 2y = 360^\circ$$

$$x + y = z + w = 180^\circ$$

This gives that

$$\angle BOC + \angle COA = 180^\circ$$

$$\angle AOD + \angle DOB = 180^\circ$$

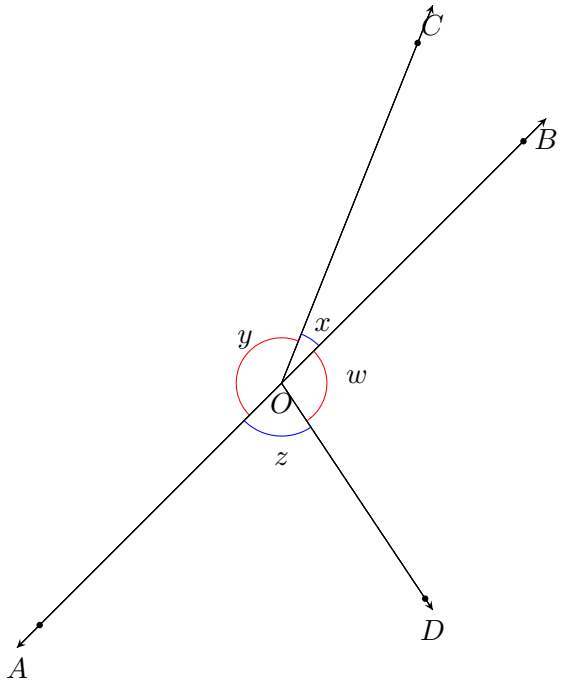



Figure 6.3.4

Which are adjacent angle, so by axiom (6.2) AOB is a line.

**Problem 6.3.5.** In Fig.(6.3.5), POQ is a line. Ray OR is perpendicular to line PQ. OS is another ray lying betweenrays OP and OR. Prove that

$$\angle ROS = \frac{1}{2}\left(\angle QOS\angle - POS\right).$$

**Solution :** In Fig.(6.3.5), ray OR stands on line POQ. So,

$$\angle QOR + \angle ROP = 180^\circ$$

(6.3.1) $$\angle QOP = \angle ROP = 90^\circ$$

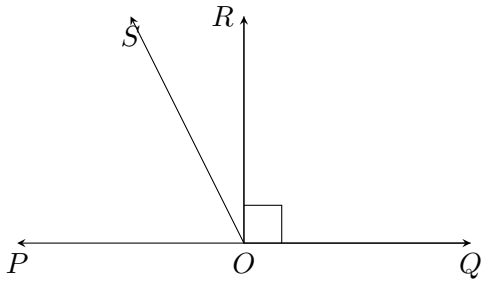



Figure 6.3.5

because ray OR $\perp$ line POQ. From Fig. (6.3.5),

$$\angle ROS = \angle SOQ - \angle ROQ \tag{6.3.2}$$

$$\angle ROS = \angle POR - \angle POS \tag{6.3.3}$$

From equation (6.3.1), (6.3.2) and (6.3.3),

$$\angle ROS = \frac{1}{2}\left(\angle QOS\angle - POS\right).$$

**Problem 6.3.6.** it is given that $\angle XYZ = 64^\circ$ and XY is produced to point P. Draw a figure from the given information. If YQ bisect $\angle ZYP$, find $\angle XYQ$ and refex $\angle QYP$.

**Solution :** In Fig.(6.3.6), ray YZ stands on line XYP. So,

$$\angle ZYP + \angle ZYX = 180^\circ$$

We have $\angle ZYX = 64^\circ$,

$$\angle ZYP = 180^\circ - 64^\circ = 116^\circ$$

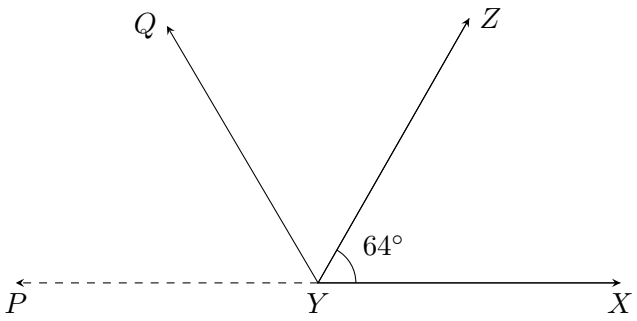



Figure 6.3.6

$$\angle XYQ = \frac{1}{2} \times \angle ZYP + \angle ZYX = \frac{1}{2} \times 116^\circ + 64^\circ = 122^\circ$$

Now $\angle QYP = \frac{1}{2} \times 116^\circ = 58^\circ$, so reflex

$$\angle QYP = 360^\circ - 58^\circ = 302^\circ$$

## 6.4 Lines Parallel to the Same Line

If two lines are parallel to the same line, then they are parallel to each other.

**Theorem 6.4.1:** *Lines which are parallel to the same line are parallel to each other.*

**Remark:** This theorem can extended more than two lines.

**Example 6.4.2.** In Fig.(6.4.1), if $PQ \parallel RS$, $\angle MXQ = 135^\circ$ and $\angle MYR = 40^\circ$, find $\angle XMY$.

**Solution :** In the Fig. (6.4.1), $PQ \parallel RS$, $\angle MXQ = 135^\circ$ and $\angle MYR = 40^\circ$. To find, $\angle XMY$. You construct a line AMB which is parallel to line PQ.

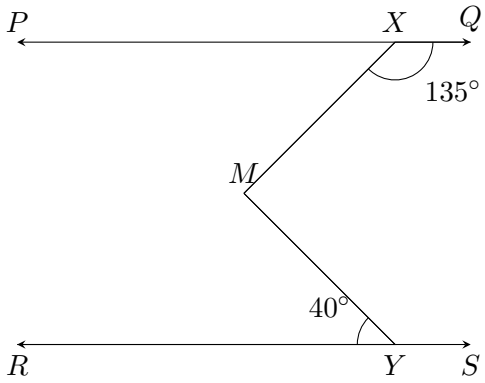



Figure 6.4.1

Now in Fig. (6.4.2), $PQ \parallel AB$ and $PQ \parallel RS$. Therefore, $AB \parallel RS$. Hence,

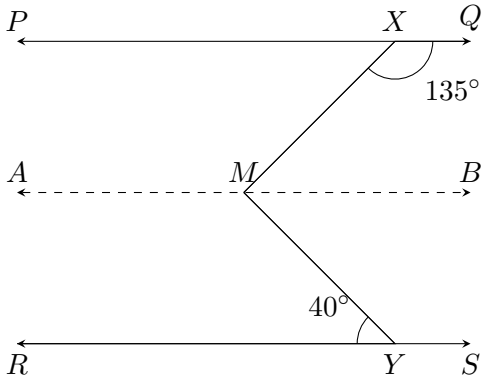



Figure 6.4.2

$$\angle QXM + \angle XMB = 180^\circ$$

($AB \parallel PQ$, interior angles on the same side of the transversal XM )

But

$$\angle QXM = 135^\circ$$

So,

$$\angle XMB = 180^\circ - 135^\circ = 45^\circ$$

$$\angle XMB = 45^\circ \tag{6.4.1}$$

Now,

$$\angle BMY = \angle MYR$$

($AB \parallel RS$, alternate angles) therefore,

$$\angle BMY = 40^\circ$$

$$\angle BMY = 40^\circ \tag{6.4.2}$$

From equation (6.4.1) and (6.4.2), you get

$$\angle XMB + \angle BMY = 45^\circ + 40^\circ$$

That is,

$$\angle XMY = 85^\circ$$

**Example 6.4.3.** If the transversal intersects two lines such that the bisectors of a pair of corresponding angles are parallel, then prove that the two lines are parallel.

**Solution :** In Fig. (6.4.3), a transversal AD intersects two lines PQ and RS at point B and C respectively. Ray BE is the bisector of $\angle QBA$ and ray

CG is the bisector of $\angle SCB$; $CG \parallel BE$. Prove that $PQ \parallel RS$. It is given that ray BE is bisector of $\angle ABQ$ Therefore,

$$\angle EBA = \frac{1}{2}\angle QBA \tag{6.4.3}$$

Similarly, ray CG is the bisector of $\angle SCB$

Therefore,

$$\angle GCB = \frac{1}{2}\angle SCB \tag{6.4.4}$$

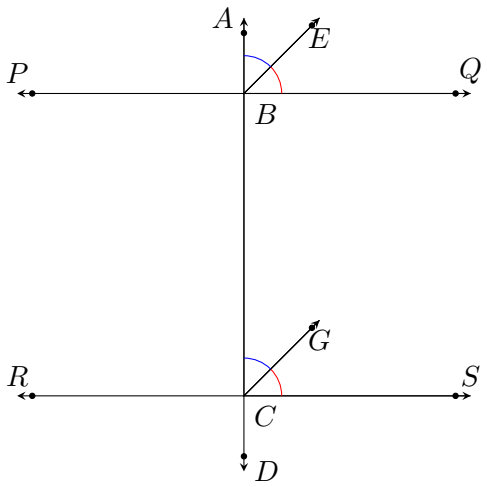



Figure 6.4.3

But, $CG \parallel BE$ and AD is transversal. Therefore,

$$\angle EBA = \angle GCB \tag{6.4.5}$$

(Corresponding angle axiom)

From Equation (6.4.3) and (6.4.4) with (6.4.5), we get

$$\frac{1}{2}\angle QBA = \frac{1}{2}\angle SCB$$

$$\angle SCB = \angle QBA$$

But the $\angle SCB$ and $\angle QBA$ are corresponding angles formed by transversal line AD with PQ and RS and equal.

Therefore,

$$RS \parallel PQ$$

(By converse of corresponding angle axiom)

**Example 6.4.4.** In Fig. (6.4.4), $AB \parallel CD$ and $CD \parallel EF$. Also $EA \perp AB$. If $\angle BEF = 55°$. find the value of $x$ $y$ and $z$.

**Solution :** In Fig. (6.4.4), In Fig. (6.4.4), $AB \parallel CD$ and $CD \parallel EF$. Also $EA \perp AB$. If $\angle BEF = 55°$.

The transversal line BE form interior angle $\angle DEF$ and $\angle GDE$ at same side. Therefore,

$$y + 55° = 180°$$

$$y = 180° - 55° = 125°$$

Next,

$$x = y$$

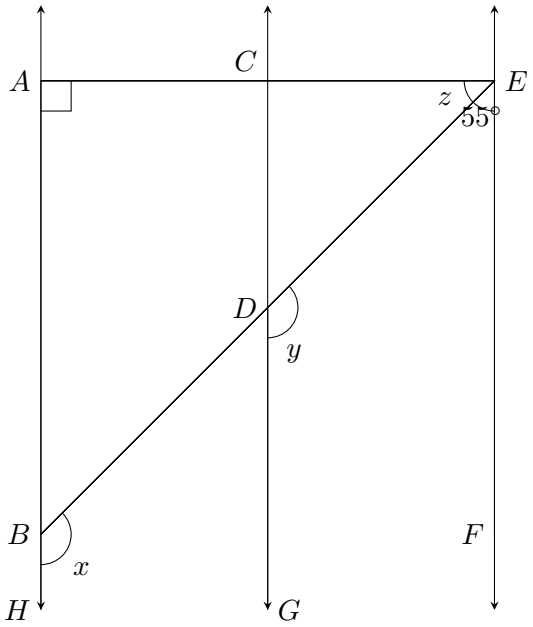



Figure 6.4.4

because they are corresponding angle. Therefore,

$$x = 125^\circ$$

So,

$$\angle AED + \angle DEF = 90^\circ$$

$$z = 90^\circ - 55^\circ = 35^\circ$$

## 6.5 Solution of Exercise 6.2

**Problem 6.5.1.** In Fig. (6.5.1), if $AB \parallel CD$, $CD \parallel EF$ and $y : z = 3 : 7$, find $x$

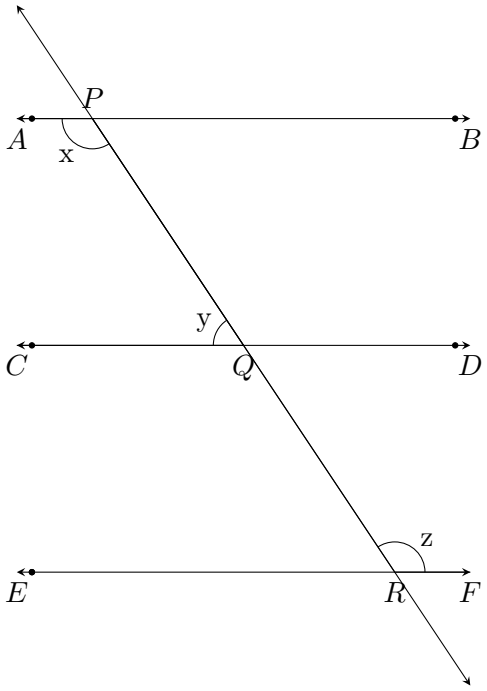



Figure 6.5.1

**Solution :** In Fig. (6.5.1), the line $AB \parallel CD$, $CD \parallel EF$ and $y : z = 3 : 7$.

The transversal line PR intresect line AB, CD and EF.

So,

$$\angle DQP = \angle FRQ$$

$$\angle DQP = z$$

This give, $y + z = 180^\circ$ (linear pair angle)

So,

$$y = \frac{3}{10} \times 180^\circ = 54^\circ$$

$$y = \frac{7}{10} \times 180^\circ = 126^\circ$$

Now $x = z = 126^\circ$ (internal alternate angle )

**Problem 6.5.2.** In Fig. (6.5.2), if $AB \parallel CD$, $EF \perp CD$ and $\angle GED = 126^\circ$, find $\angle AGE$, $\angle GEF$ and $\angle FGE$.

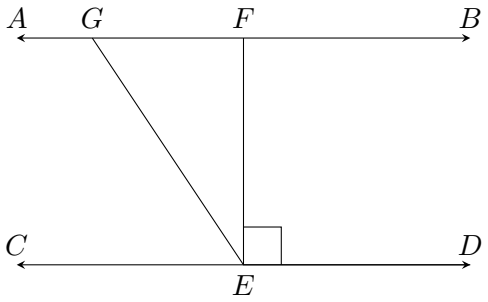



Figure 6.5.2

**Solution :** In Fig. (6.5.2), the line $AB \parallel CD$, and $EF \perp CD$. So, $EF \perp AB$ and $\angle GFE = 90^\circ$. Given that, $\angle GED = 126^\circ$. In the figure

$$\angle GEF + \angle FED = \angle GED = 126^\circ$$

$$\angle GEF = 126^\circ - 90^\circ = 36^\circ$$

In $\triangle GEF$,

$$\angle EGF = 180^\circ - 36^\circ - 90^\circ = 54^\circ$$

$$\angle AGE = \angle GED = 126^\circ$$

(Corresponding angle)

**Problem 6.5.3.** In Fig. (6.5.3), if $PQ \parallel ST$, $\angle PQR = 110^\circ$ and $\angle RST = 130^\circ$, find $\angle QRS$.

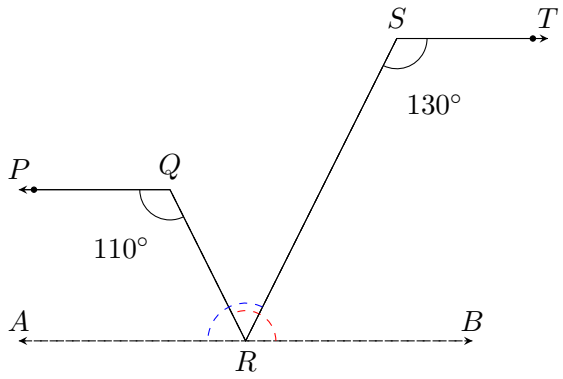



Figure 6.5.3

**Solution :** Draw a line parallel to $ST$ through point R. In the Fig. (6.5.3). Now in Fig. (6.5.4), $PQ \parallel ST$, $AB \parallel ST$ and $\angle PQR = 110^\circ$ and $\angle RST = 130^\circ$

So,

$$\angle QRB = \angle PQR$$

(Alternate angle axiom)

$$\angle SRA = \angle RSA$$

(Alternate angle axiom)

In the Fig. (6.5.3),

$$\angle ARS + \angle QRB = 180^\circ + \angle QRS$$

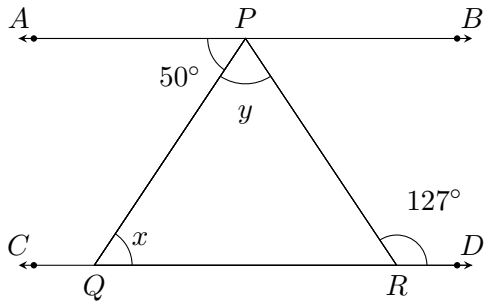



Figure 6.5.4

$$\angle QRS = 130^\circ + 110^\circ - 180^\circ$$

$$\angle QRS = 60^\circ$$

**Problem 6.5.4.** In Fig. (6.5.4), if $AB \parallel CD$, $\angle APQ = 50^\circ$ and $\angle PRD = 127^\circ$, find $x$ and $y$.

**Solution :** In Fig. (6.5.4), The line $AB \parallel CD$ and PQ is transversal line segment, so

$$\angle APQ = \angle PQR$$

(Alternate angle axiom)

$$x = 50^\circ$$

Next $AB \parallel CD$ and PR is transversa;l line. So,

$$\angle APR = \angle PRD$$

$$\angle APQ + \angle QPR = 127^\circ$$

$$y = 127^\circ - 50^\circ = 77^\circ$$

**Problem 6.5.5.** In Fig. (6.5.5), PQ and RS are two mirrors placed parallel to each other. An incident ray AB strikes the mirror PQ at B, the reflected ray moves along the path BC and strikes the mirror RS at C and again reflects back along CD. Prove that $AB \parallel CD$

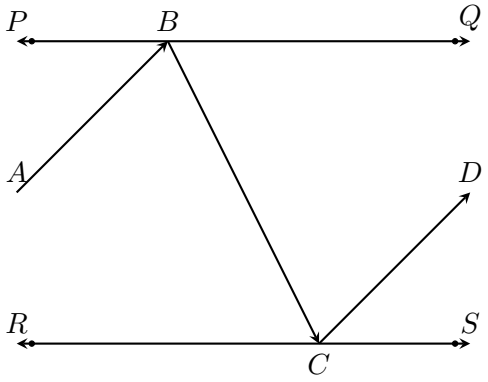



Figure 6.5.5

*Proof.* Construct the mormal BE and CF at the points B and C in Fig. (6.5.5). Now in Fig. (6.5.6), AB is incident ray and BC is reflected ray at point B on mirror PQ. So,

(6.5.1) $$\angle ABE = \angle EBC$$

(By law of reflection.)

BC is incident ray and CD is reflected ray at point C on mirror RS. So,

(6.5.2) $$\angle BCF = \angle FCD$$

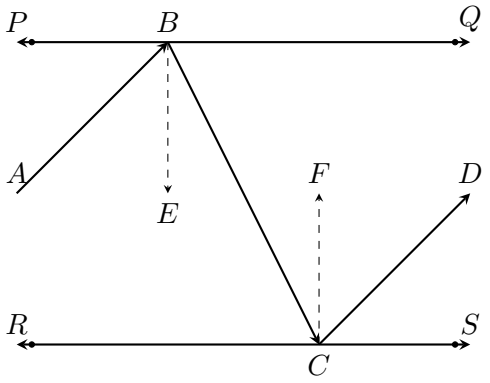



Figure 6.5.6

(By law of reflection.)

The normal $BE \parallel FC$, because they are mornal on parallel line PQ and RS.

The line BC is transversal line.

Therefore,

$$\angle EBC = \angle BCF \tag{6.5.3}$$

(alternate angle axiom) You get

$$2\angle EBC = 2\angle BCF$$

From equation (6.5.1), (6.5.2) and (6.5.3),

$$\angle ABC = \angle BCD \tag{6.5.4}$$

Hence,

$$AB \parallel CD.$$

( Reverse alternate angle axiom.) □

# 7 TRIANGLES

## 7.1 Introduction

You will study the planar constructed through three intersecting lines. For this shape we use terminology **‘ triangle’**. A triangle has six part three angles, three sides and three vertices. In previous classes you have been studied many relation and properties between the parts of triangle but in the present chapter some of them.

## 7.2 Congruence of Triangles

In your daily life you observed that any page of your book, all mobiles of same company and model, two ATM cards issued by the same bank are identical. These figure are equal in all aspects or these figures are equal in shape and sizes. Such figuers are called **congruent figures**.

Now, if draw two circles of the same radius and place one on the other. You will see that they cover each other completely and you call them as congruent circles. You think if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle, then their crresponding vertices will be same positions. Hence the all parts of one triangle are equal to the all correponding part of the other triangle. These two triangles are

**'congruent triangle'**.

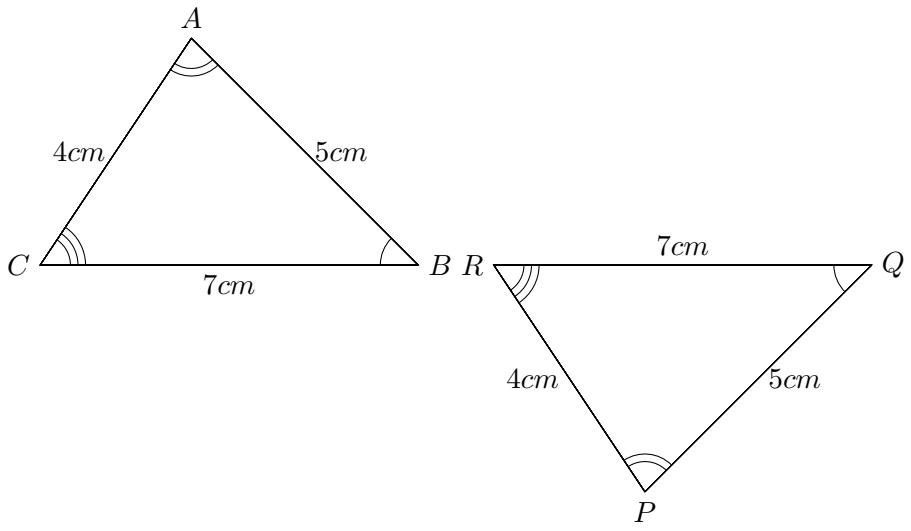



Figure 7.2.1

In the given Fig. (7.2.1), $\triangle ABC$ and $\triangle PQR$, PQ covers AB, QR covers BC, RP covers CA; $\angle Q$ covers $\angle B$, $\angle P$ covers $\angle A$, $\angle R$ covers $\angle C$, P correspond to A, Q correspond to B, R correspond to C, that is $P \Leftrightarrow A$, $Q \Leftrightarrow B$, $R \Leftrightarrow C$.

Hence $\triangle ABC$ and $\triangle PQR$ congruent, symbolicaly it is writen as

$$\triangle ABC \cong \triangle PQR$$

*Remark* 7.2.1. $\triangle ABC \cong \triangle PQR \not\Rightarrow \triangle ABC \cong \triangle QRP$

In congruent triangles corresponding parts are equal and it is written in

short " CPCT" for *corresponding part of congruent triangles.*

## 7.3 Criteria for Congruence of Triangles

The criteria for congruence of triangles are that conditions through them unique triangle can be form.

**Definition 7.3.1** (Axiom SAS congruence rule)**.** Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.

**Example 7.3.2.** In Fig. (7.3.1), $OA = OB$ and $OD = OC$. Show that '

(i) $\triangle AOD \cong \triangle BOC$

(ii) $AD \parallel BC$

**Solution :** (i) In the Fig. (7.3.1), consider $\triangle AOD$ and $\triangle BOC$, given that,

$OA = OB \quad OD = OC$.

$\angle BOC$ and $\angle AOD$ are vertically opposite angles, you get

$$\angle BOC = \angle AOD$$

So,

$$\triangle BOC \cong \triangle AOD$$

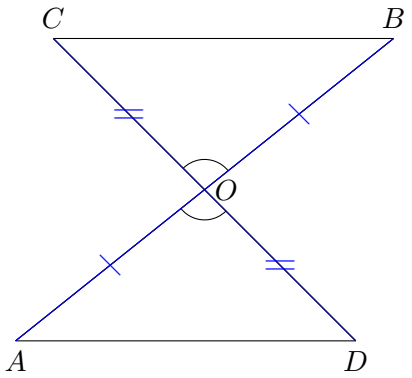



Figure 7.3.1

(SAS congruence axiom)

(ii) In congruent $\triangle AOD$ and $\triangle BOC$, the other corresponding parts are also equal.

So,

$$\angle OCB = \angle ODA$$

These angle are alternate angle for line segment AD and BC. So, by reverse alternate angle axiom,

$$AD \parallel BC.$$

**Example 7.3.3.** AB is a line segment and line $l$ is its perpendicular bisector. If a point P lies on $l$, show that P is equidistant from A and B.

**Solution :** In the Fig. (7.3.2), line $l \perp AB$ and bisected at point C. So, C is the mid-point of AB.

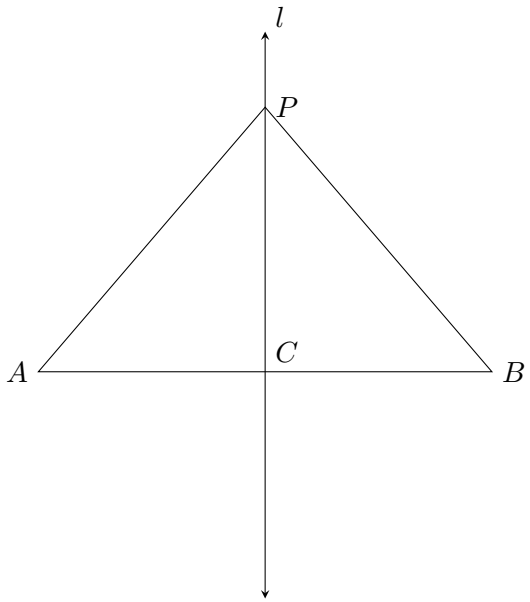



Figure 7.3.2

Consider $\triangle PAC$ and $\triangle PCB$

$$AC = BC$$

( C is the mid-point of AB)

$$\angle PCA = \angle PCB = 90^\circ$$

(line $l$ is perpendicular bisector of AB.)

$$PC = PC$$

(Common sides of triangle.)

So,

$$\triangle PAC \cong \triangle PCB$$

(SAS axiom)

Hence, $PA = PB$, because they are corresponding sides of congruent triangles.

*Remark* 7.3.4. It is very important that the equal angles are included between the pair of equal sides. So, SAS congruence axiom holds but ASS or SSA not axiom.

**Theorem 7.3.5(ASA congruence rule):** *Two triangles are congruent if two angles and the included side of one triangle are equal to two angle and included side of other triangle.*

*Proof.* We are given two $\triangle ABC$ and $\triangle DEF$ in which

$$\angle B = \angle E, \quad \angle C = \angle F$$

and

$$BC = EF$$

We need to prove that

$$\triangle ABC \cong \triangle DEF$$

For proving the congruence of the two triangles see the cases aries.

**Case (i) :** Consider the Fig. (7.3.3). Let $AB = DE$

Think as,

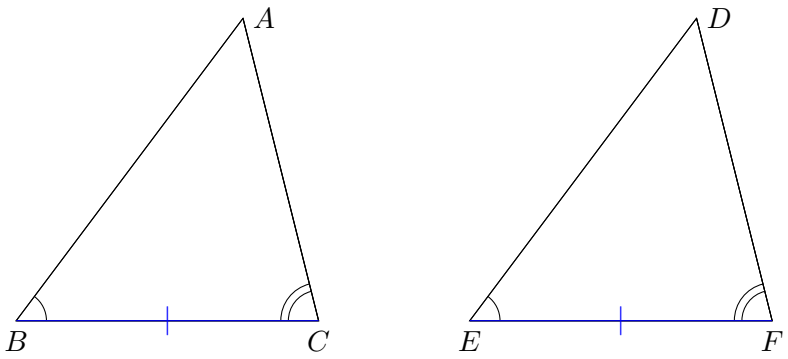



Figure 7.3.3

$AB = DE$ (Assumed), $\angle B = \angle E$ (Given),

$BC = EF$ (Given), So,

$$\triangle ABC \cong \triangle DEF$$

(SAS axiom)

**Case (ii) :** Let if possible $AB < DE$. Then you can take a point P on DE such that $AB = EP$. Now consider $\triangle ABC$ and $\triangle PEF$ as in Fig. (7.3.4),

Think that in $\triangle ABC$ and $\triangle PEF$,

$AB = PE$ (By construction), $\angle B = \angle E$ (Given),

$BC = EF$ (Given). so, on the above fact,

$\triangle ABC \cong \triangle PEF$ ( By SAS axiom) When two triangles are congruent then corresponding parts will be equal.

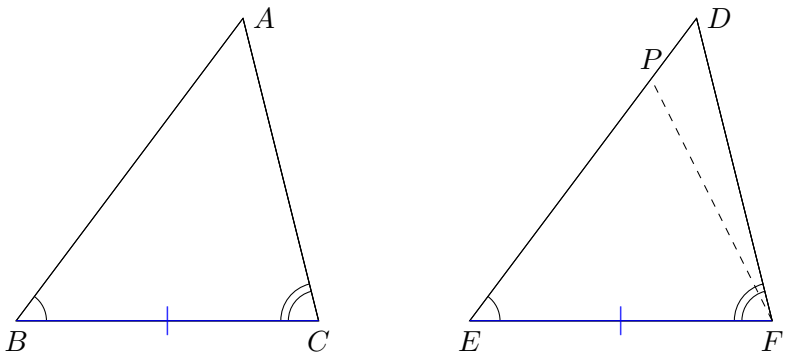



Figure 7.3.4

So,

$$\angle ACB = \angle PFE$$

But we are given that

$$\angle ACB = \angle DFE$$

So,

$$\angle PFE = \angle DFE$$

This possible only if P coincides with D. It means $AB = ED$. Hence $\triangle ABC \cong \triangle DEF$.

**Case (iii) :** if $AB > DE$, We can take a point H on AB such that $BH = ED$. So, similar logic of case (ii), we can conclude that $AB = DE$ and so, $\triangle ABC \cong \triangle DEF$. □

**Example 7.3.6.** Line-segment AB is parralel to another line-segment CD. O is the mid-point of AD. show that (i) $\triangle AOB \cong \triangle DOC$ (ii) O is also the mid-point of BC.

**Solution :** (i) In Fig.(7.3.5), consider $\triangle AOB$ and $\triangle DOC$,

$\angle AOB = \angle DOC$ (Vertically opposite angles)

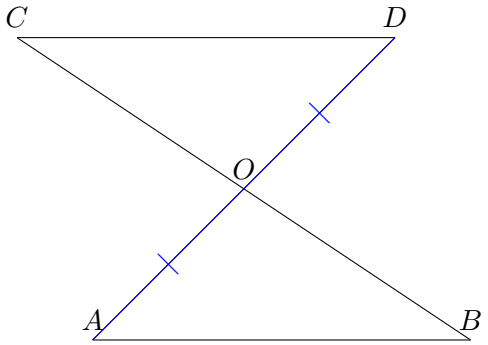



Figure 7.3.5

$\angle BAO = \angle ODC$ (alternate angle axiom)

$OA = OD$ (Given)

therefore, $\triangle AOB \cong \triangle DOC$ (ASA rule).

**Case (ii):** We have $\triangle AOB \cong \triangle DOC$

So, $OB = OC$ (By CPCT), Hence O is the mid-point of BC.

## 7.4 Solution of Exercise 7.1

**Problem 7.4.1.** In Fig. (7.4.1), quadrilateral ABCD, $AC = AD$ and $AB$ bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD ?

**Solution :** In Fig. (7.4.1), the quadrilateral ABCD have sides, $AC = AD$ and $\angle BAC = \angle DAB$.

Consider $\triangle ABC$ and $\triangle ABD$

We have given that, $\angle BAC = \angle DAB$ (Given that line AB bisects $\angle A$),

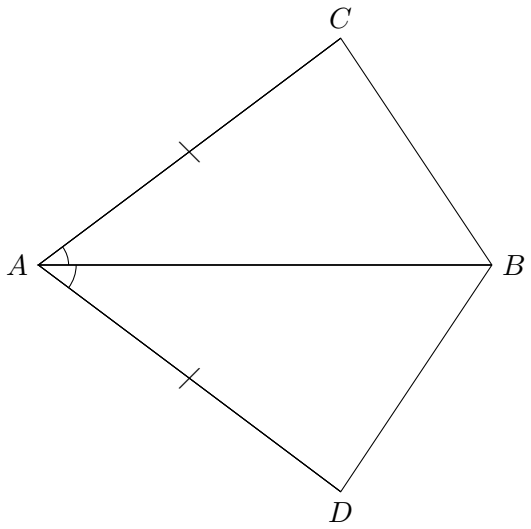



Figure 7.4.1

$AC = AD$ $(Given)$, $AB = AB$ $(Give\ common\ side)$.

Hence

$$\triangle ABC \cong \triangle ABD \qquad (SAS \text{ congruence rule})$$

$$BC = BD \qquad (ByCPCT)$$

**Problem 7.4.2.** In Fig. (7.4.2), ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that,

i $\triangle ABD \cong \triangle BAC$

ii $BD = AC$

iii $\angle ABD = \angle BAC$

**Solution :** In Fig. (7.4.2), consider $\triangle BAD$ and $\triangle CBA$

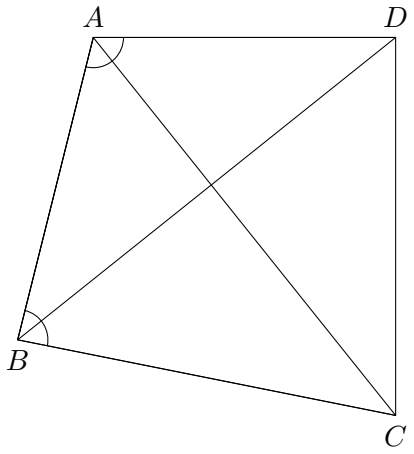



Figure 7.4.2

$$AD = BC \qquad (Given)$$

$$AB = AB \qquad (Common\ side)$$

$$\angle DAB = \angle CBA \qquad (Given)$$

Hence

$$\triangle ABD \cong \triangle BAC \qquad (By\ SAS\ axiom).$$

So, $BD = AC$ and $\angle ABD = \angle BAC$, (By CPCT).

**Problem 7.4.3.** In Fig. (7.4.3), AD and BC are equal perpendicualrs to a line segment AB. Show that CD bisects AB.

**Solution :** In Fig. (7.4.3), the line $BC \perp BA$ and $DA \perp AB$. Therefore, line $BC \parallel AD$. Consider $\triangle BCO$ and $\triangle ADO$.

$$\angle CBO = \angle DAO = 90^\circ$$

$$\angle OCB = \angle ODA \quad (Alternate\ angle\ axiom)$$

$$CB = DA \quad (Given)$$

Hence,

$$\triangle BCO \cong \triangle ADO$$

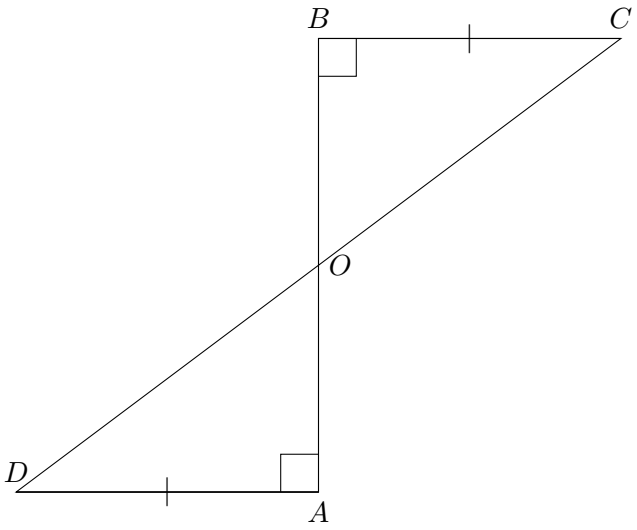



Figure 7.4.3

Now, we get

$$OB = OA \qquad (By\ CPCT)$$

. We get conclusion that CD bisects AB.

**Problem 7.4.4.** $l$ and $m$ are two parallel lines intersected by another pair of parallel lines $p$ and $q$. Show that $\triangle ABC \cong \triangle CDA$.

In the Fig. (7.4.4), the line-segment AC intersects the parallel lines $l$ and $m$. So, $\angle DAC = \angle ACB$. The line-segment AC intersects to parallel lines $p$ and $q$. So, $\angle DCA = \angle CAB$.

Consider the $\triangle ADC$ and $\triangle CAB$,

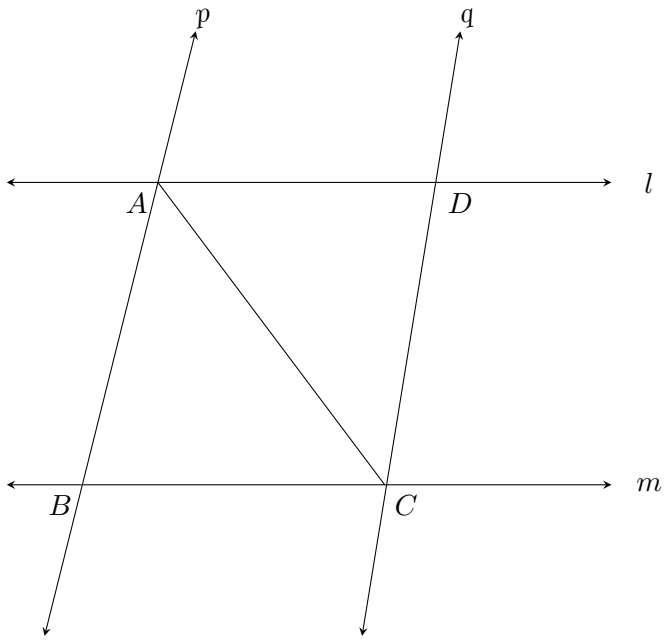



Figure 7.4.4

$$\angle DAC = \angle ACB \qquad \textit{Alternate angle axiom}$$

$$\angle DCA = \angle CAB \qquad \textit{Alternate angle axiom}$$

The side AC is common in bothe $\triangle ADC$ and $\triangle CAB$. So,

$$\triangle ABC \cong \triangle CDA.$$

(by ASA rule)

**Problem 7.4.5.** line $l$ is the bisector of an angle $\angle A$ and B is any point on $l$. BP and BQ are perpendiculars from B to the arms of $\angle A$ in Fig. (7.4.5).

Show that :

i $\triangle APB \cong \triangle AQB$

ii $BP = BQ$ or B is equidistant from the arms of $\angle A$.

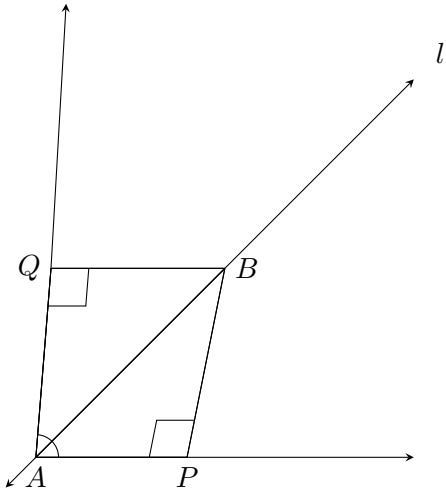



Figure 7.4.5

**Solution :** In Fig. (7.4.5), line $l$ is angle bisector of $\angle A$ and $QA \perp QB$ and $AP \perp PB$.

In $\triangle ABQ$ and $\triangle ABP$.

$$\angle BAQ = \angle BAP$$

(Line $l$ is angle bisector of $\angle QAP$)

$$\angle BQA = \angle BPA, \quad AQ \perp QB, \quad AP \perp BP$$

The line-segment AP is common side. Hence

$$\triangle ABQ \cong \triangle ABP \qquad (By\ AAS\ rule)$$

So, $BQ = BP$ (By CPCT).

**Problem 7.4.6.** In Fig. (7.4.6), $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.

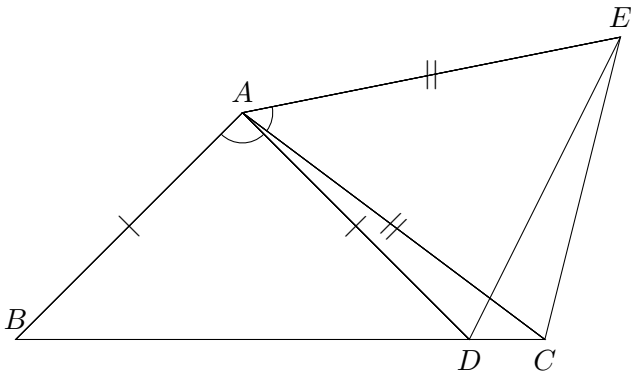



Figure 7.4.6

**Solution :** In Fig. (7.4.6), consider $\triangle ABC$ and $\triangle ADE$

$$AB = AD \qquad (Given)$$

$$AC = AE \qquad (Given)$$

$$\angle BAD = \angle EAC \qquad (Given)$$

So,

$$\triangle ABC \cong \triangle ADE \qquad (by\ SAS\ axiom)$$

Therefore, $BC = DE$, by (CPCT)

**Problem 7.4.7.** AB is a line segment and P is its mid-point. D and E are points on the same side of AB such that $\angle BAD = \angle ABE$ and $\angle EPA = \angle DPB$ according Fig. (7.4.7). Show that,

i $\triangle DAP \cong \triangle EBP$

ii $AD = BE$

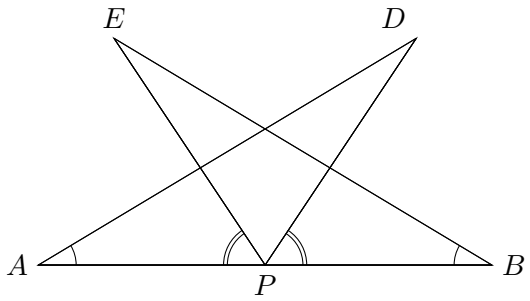



Figure 7.4.7

**Solution :** In Fig. (7.4.7), point P is mid point of line segment AB, so $AP = BP$. Now in $\triangle DAP$ and $\triangle EBP$,

$$\angle BAD = \angle ABE \qquad (Given)$$

$$\angle EPA = \angle DPB \qquad (Given)$$

$$PA = PB$$

Therefore,

$$\triangle DAP \cong \triangle EBP \qquad (ASA\ rule)$$

.

Hence,

$$AD = BE \qquad (CPCT)$$

**Problem 7.4.8.** In right angle $\triangle ABC$, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to B (as in Fig. (7.4.8)). Show that:

i $\triangle AMC \cong \triangle BMD$

ii $\triangle DBC$ is a right angle.

iii $\triangle DBC \cong \triangle ACB$

iv $CM = \frac{1}{2}AB$

**Solution :** In Fig. (7.4.8), M is the mid-point of AB, and $C$ is a right angle and DM = CM. Consider in $\triangle AMC$ and $\triangle BMD$

$$AM = BM \qquad (Given)$$

$$DM = MC \qquad (Given)$$

$$\angle AMC = \angle BMD \qquad (Vertically\ opposite\ angle)$$

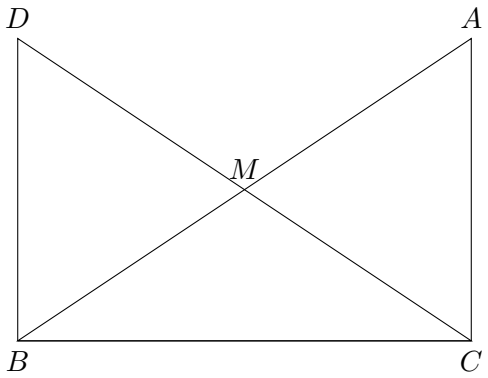



Figure 7.4.8

Hence

$$\triangle AMC \cong \triangle BMD \qquad (SAS\ axiomn)$$

Now $AC = BD$ and $\angle ACB = \angle DBC$ (By CPCT)

But $\angle BCA = 90^\circ$. So,

$$\angle DBC = 90^\circ$$

Consider in $\triangle DBC$ and $\triangle ACB$

$$AC = BD \qquad (Given)$$

$$BC = BC \qquad (Common)$$

$$\angle ACB = \angle DBC = 90^\circ \qquad (Given)$$

Hence

$$\triangle DBC \cong \triangle ACB \qquad (SAS\ axiom)$$

Therefore, $AB = DC$ (By CPCT).

$$DC = 2MC = AB \implies MC = \frac{1}{2}AB.$$

## 7.5 Some Properties of a Triangle

In the section (7.4), we have studied two criteria for congruence of a triangle. In these two criteria, one is axiom and other is theorem. Now we find out results apply this criteria of congruence of triangles.

**Theorem 7.5.1:** *Angles opposite to equal sides of an isosceles triangle are equal.*

*Proof.* We are given an isosceles triangle ABC in which $AB = AC$. We need to prove that $\angle B = \angle C$.

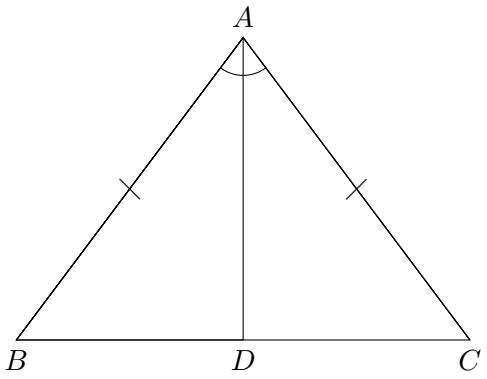



Figure 7.5.1

Let us draw angle bisector of $\angle A$ which meets at point D, on BC. So, $\angle BAD = \angle CAD$.

Consider in $\triangle BAD$ and $\triangle CAD$

$$AB = AC \qquad (Given)$$

$$\angle BAD = \angle CAD \qquad (By\ construction)$$

$$AD = AD \qquad (Commmon)$$

So,

$$\triangle BAD \cong \triangle CAD \qquad (By\ SAS\ axiom)$$

so, $\angle ABD = \angle ACD$, since they are corresponding angles of congruent triangles. So,

$$\angle B = \angle C.$$

□

**Theorem 7.5.2:** *The sides opposite to equal angle of an isosceles triangle are equal.*

*Proof.* We are given an isosceles triangle ABC in which$\angle B = \angle C$. We need to prove that $AB = AC$.

Let us draw angle bisector of $\angle A$ which meets at point D, on BC. So, $\angle BAD = \angle CAD$.

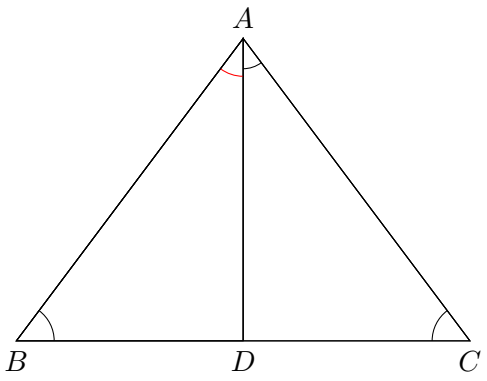



Figure 7.5.2

Consider in $\triangle BAD$ and $\triangle CAD$

$$\angle B = \angle C \qquad (Given)$$

$$\angle BAD = \angle CAD \qquad (By, construction)$$

Therefore

$$\angle ADB = \angle ADC \quad (Triangle\ properties)$$

$$AD = AD \qquad (Commmon)$$

So,

$$\triangle BAD \cong \triangle CAD \qquad (By\ ASA\ rule)$$

so,

$$AB = AC.$$

Since they are corresponding sides of congruent triangles. So,

$$AB = AC.$$

□

**Example 7.5.3.** In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC (given Fig. 7.5.3). Show that $AB = AC$ and $\triangle ABC$ is isosceles.

**Solution :**

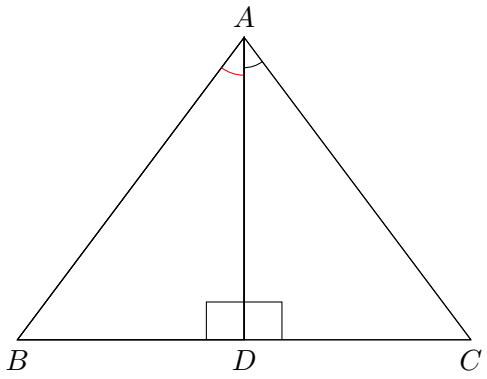



Figure 7.5.3

Consider in $\triangle ADB$ and $\triangle ADC$.

$$\angle DAB = \angle DAC \qquad (Given)$$

$$\angle ADB = \angle ADC = 90^\circ \qquad (Given)$$

$$DA = DA \qquad (Common\ sides)$$

So,

$$\triangle ABD \cong \triangle ADC \qquad (By\ ASA\ rule)$$

Hence,

$$AB = AC \qquad (CPCT)$$

Therefore, $\triangle ABC$ is an isosceles triangle.

**Example 7.5.4.** E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$ Given in Fig, (7.5.4). Show that $BF = CE$.

**Solution :** In Fig.(7.5.4), in $\triangle ABC$, the equal sides AB and AC have mid-points E and F. So, $\angle ABC = \angle ACB$, which means $\angle EBC = \angle FCB$.

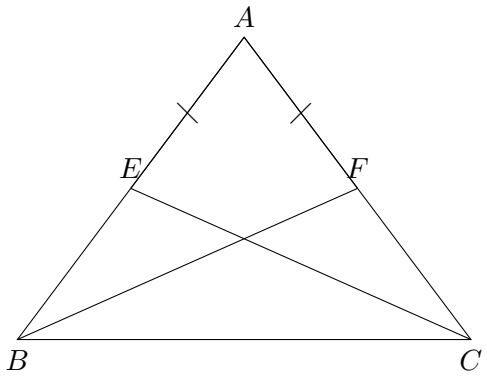



Figure 7.5.4

Now, in $\triangle BEC$ and $\triangle BFC$,

$$\angle EBC = \angle FCB \qquad (Angle\ of\ isosceles\ triangle)$$

$$EB = FC \qquad (Half\ side\ of\ isosceles\ triangle)$$

$$BC = CB \qquad (Common\ side)$$

So,

$$\triangle BEF \triangle CFB \qquad (SAS\ axiom)$$

Therefore,

$$EC = BF \qquad (CPCT)$$

**Example 7.5.5.** In Fig.(7.5.5), an isosceles $\triangle ABC$ with $AB = AC$, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.

**Solution :** In $\triangle ABE$ and $\triangle ACD$

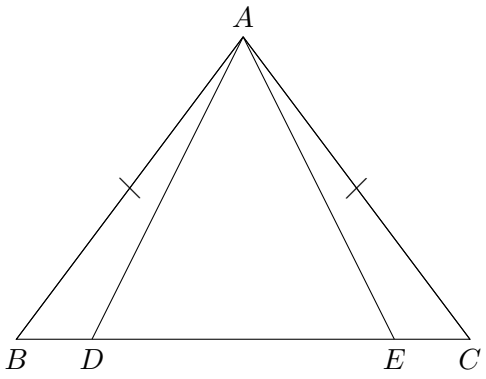



Figure 7.5.5

$$AB = AC \qquad (Given)$$

$$BE = CD \qquad (Given)$$

$$\angle B = \angle C \qquad (Angle\ opposite\ to\ equal\ sides)$$

So,

$$\triangle ABE \cong \triangle ACD \qquad (SAS, axiom)$$

Hence,

$$AE = AD \qquad (CPCT)$$

## 7.6 Solution of Exercise 7.2

**Problem 7.6.1.** In an isosceles $\triangle ABC$, with $AB = AC$, the bisector of $\angle B$ and $\angle C$ intersect each at O. Join A to O. Show that : (i) $OB = OC$ (ii) AO bisect $\angle A$.

**Solution :** In Fig. (7.6.1), take $\triangle AEB$ and $\triangle ADC$

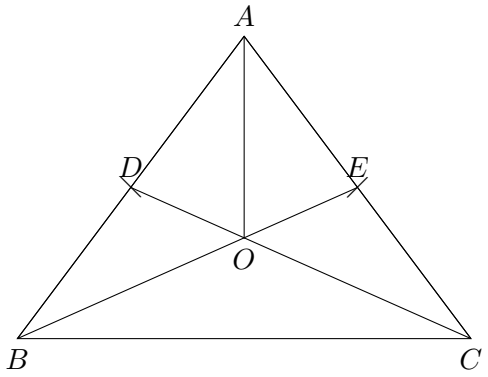



Figure 7.6.1

$$\angle ABE = \angle ACD \qquad (half\ angle\ of\ opposite\ of\ equal\ sides)$$

$$AB = AC \qquad (Given)$$

$$\angle A = \angle A \qquad (Common)$$

So.

$$\triangle ABE \cong \triangle ACD \qquad (ASA,\ rule)$$

Hence,

$$AD = AE \qquad (CPCT)$$

Now, in $\triangle DOB$ and $\triangle EOC$

$$DB = EC \qquad (Because\ AB = AC)$$

$$\angle DBO = \angle ECO \qquad (\angle B = \angle C)$$

$$\angle DOB = \angle EOC \qquad (Vertically\ opposite\ angle)$$

Hence,

$$\triangle DOB \cong \triangle EOC$$

$$OB = OC \qquad (CPCT)$$

and

$$OD = OE \qquad (CPCT)$$

Now, in $\triangle AOD$ and $\triangle AOE$,

$$ADAE \qquad (because\ AB = AC)$$

$$OD = OE \qquad (\triangle DOB \cong \triangle EOC)$$

$$\angle ADO = \angle AEO \qquad (\triangle ABE \cong \triangle ACD)$$

Therefore,

$$\triangle AOE \cong \triangle AOD$$

$$\angle DAO = \angle OAE \qquad (CPCT)$$

Hence, OA is bisector of $\angle A$

**Problem 7.6.2.** In Fig. (7.6.2), $\triangle ABC$, AD is perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which sides $AB = AC$.

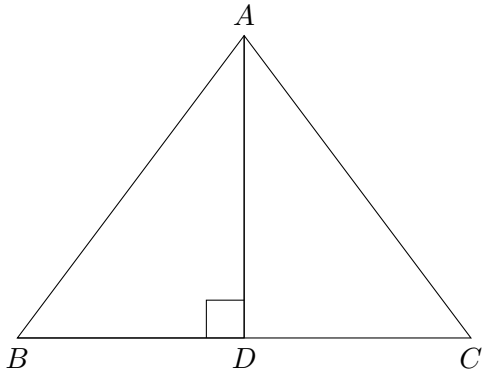



Figure 7.6.2

**Solution :** In Fig. (7.6.2), AD is perpendicular bisector of BC. So, $BD = DC$.

Now, in $\triangle ABD$ and $\triangle ADC$

$$\angle ADB = \angle ADC = 90^\circ \qquad (Linear\ pair\ angle)$$

$$DB = DC \qquad (AD \ bisect \ BC)$$

$$AD = AD \qquad (Common)$$

So,

$$\triangle ADB \cong \triangle ADC \qquad (SAS \ axiom)$$

Hence,

$$AB = AC \qquad (CPCT)$$

Therefore, $\triangle ABC$ is isosceles triangle.

**Problem 7.6.3.** In Fig. (6.7.3), ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AB and AC respectively. Show that these altitedes are equal.

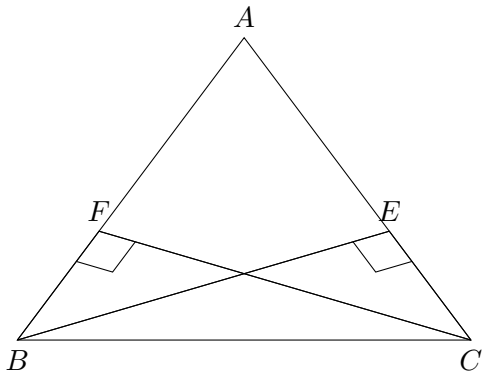



Figure 7.6.3

**Solution :** In Fig. (7.6.3), $\triangle ABC$ is isosceles triangle which side $AB = AC$.

Cosider in $\triangle BFC$ and $\triangle CEB$,

$$\angle FBC = \angle ECB \qquad (Opposite\ angle\ of\ equal\ sides)$$

$$\angle BFC = \angle CEB = 90^\circ \qquad (BE \perp AC,\ CF \perp AB)$$

$$BC = BC \qquad (Common)$$

So,

$$\triangle BFC \cong \triangle CEB \qquad (AAS\ rule)$$

Hence,

$$CFEB \qquad (CPCT)$$

**Problem 7.6.4.** ABC is a triangle in which altitude BE and CF to sides AC and AB are equal as in Fig. (7.6.4). Show that:

i $\triangle ABE = \triangle ACF$

ii $AB = AC$, i.e ABC is an isosceles triangle.

**Solution :** Cosider in $\triangle ABE$ and $\triangle ACF$,

$$\angle BFC = \angle CEB = 90^\circ \qquad (BE \perp AC,\ CF \perp AB)$$

$$BE = FC \qquad (Given)$$

$$\angle A = \angle A \qquad (Common)$$

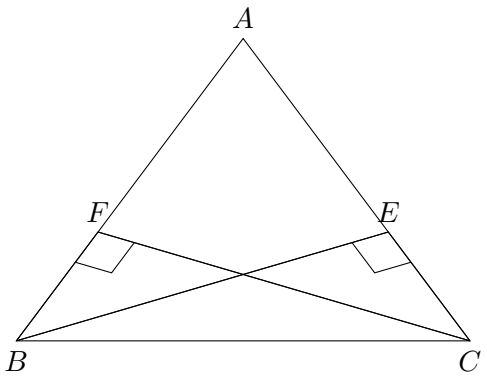



Figure 7.6.4

So,

$$\triangle ABE \cong \triangle ACF \qquad (AAS\ Rule)$$

Hence,

$$AB = AC \qquad (CPCT)$$

**Problem 7.6.5.** ABC and DBC are two isosceles triangles on the same base BC as in Fig. (7.6.5). Show that $\angle ABD = \angle ACD$.

Now, in $\triangle ABC$ the sies AB and AC are equal. So, $\angle ABC$ and $\angle ACB$ are equal angles.

(7.6.1) $$\angle ABC = \angle ACB$$

Similaraly, in $\triangle BDC$,

(7.6.2) $$\angle CBD = \angle BCD$$

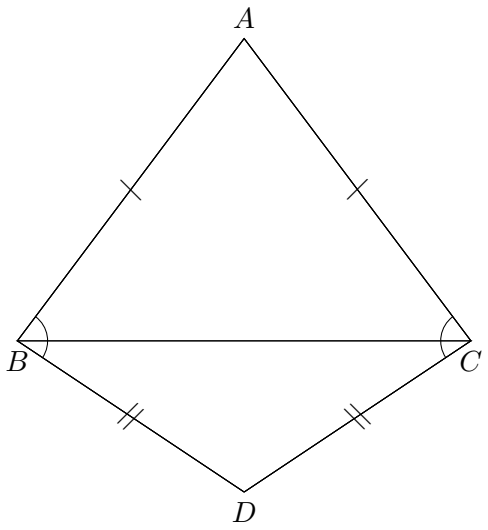



Figure 7.6.5

From, equation (7.6.1) and (6.7.2),

$$\angle ABC + \angle CBD = \angle ACB + \angle BCD \tag{7.6.3}$$

Therefore

$$\angle ABD = \angle ACD$$

**Problem 7.6.6.** $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side AB produced to D such that $AD = AB$ as in Fig. (7.6.6). Show that $\angle BCD$ is a right angle.

In Fig. (7.6.6), the $\triangle ABC$ and $\triangle ADC$ are isosceles triangle with $AB = AC$ and $AC = AD$ respectively. So,

$$\angle ABC = \angle ACB \qquad (Angle\ of\ isosceles\ triangle)$$

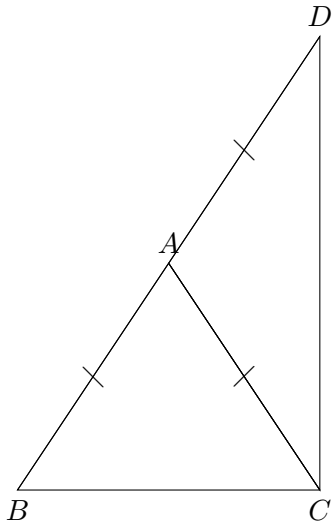



Figure 7.6.6

and

$$\angle ADC = \angle ACD \qquad (Angle\ of\ isosceles\ triangle)$$

$$\angle CAD = \angle ABC + \angle ACB \qquad (Exterior\ angle)$$

So,

$$\angle CAD = 2\angle ACB$$

Similarly,

$$\angle BAC = 2\angle ACD$$

$$\angle CAD + \angle BAC = 180^\circ \qquad (Linear\ pair\ angle)$$

$$2\angle ACB + 2\angle ACD = 180^\circ$$

$$2(\angle ACB + \angle ACD) = 180^\circ$$

Hence,

$$\angle BCD = 90^\circ$$

**Problem 7.6.7.** ABC is a right angled triangle in which $\angle A = 90^\circ$ and $AB = AC$. Find $\angle B$ and $\angle C$.

**Solution :** In Fig.(7.6.7), ABC is right angle triangle with $\angle A = 90^\circ$ and $AB = AC$.

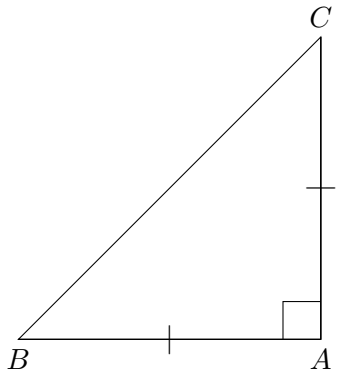



Figure 7.6.7

The $\triangle ABC$ is an isosceles triangle. Hence,

$$\angle B = \angle C \qquad (opposite\ angle\ of\ equal\ side)$$

$$\angle A + \angle B + \angle C = 180^\circ \qquad (Triangle\ properties)$$

$$\angle B + \angle C = 90^\circ$$

$$\angle B = 45^\circ = \angle C \qquad (\angle B = \angle C)$$

**Problem 7.6.8.** Show that the angles of an equilateral triangle are $60^\circ$ each.

**Solution :** In Fig. (7.6.8), $\triangle ABC$ is an equilateral triangle. Then it is isosceles triangle also.

Now, in $\triangle ABC$, $AB = AC$, So,

$$\angle C = \angle B \qquad (Opposite\ angle\ of\ equal\ side)$$

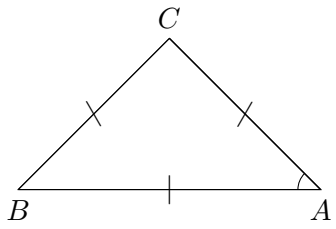



Figure 7.6.8

Similarly,

$$\angle B = \angle C \qquad (Opposite\ angle\ of\ equal\ side)$$

Therefore,

$$\angle B = \angle C = \angle A$$

$$\angle B + \angle C + \angle A = 180^\circ \qquad (triangle\ property)$$

Hence,

$$\angle B = \angle C = \angle A = 60^\circ.$$

## 7.7 Some Other criteria for Congruence of Triangle

We have seen in section (7.5) and (7.3) the property of congruence triangle but in this section discussion that we can not derived a conclusion if three angle of a triangle are equal to three angle of other triangle. What you say if three sides of a triangle are equal to three sides of other triangle.

**Theorem 7.7.1(SSS congrunce rule): :** *If three sides of one triangle are equal to the three sides of another triangle, then the two triangle are congruent.*

*Proof.* The $\triangle ABC$ and $\triangle EDF$ are congruent.

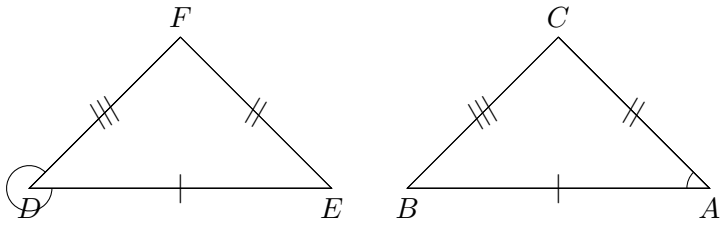



Figure 7.7.1

There are three cases aries to prove the theorem. The proof left for reader.

**Case (i)** Assumption $AB = DE, AC = EF, BC = DF$.

**Case (ii):** if $AB > DE$, then arived the conclusion that the assumption is wrong.

**Case (iii):** if $AB < DE$, then arived the conclusion that the assumption is wrong. □

**Theorem 7.7.2(RHS congruence rule):** *If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congurent.*

*Proof.* It is special case of SAS congruence axiom. Anyone easly can prove it. □

**Note :** RHS stands for **Right angle - Hypotenuse - Side.**

**Example 7.7.3.** AB is a line-segment. P and Q are points on opposite side of AB such that each of them is equidistant from the points A and B as in Fig. (7.7.2). Show that the line PQ is the perpendicular bisector of AB.

**Solution :** In Fig. (7.7.2), consider in $\triangle PAQ$ and $\triangle PBQ$

$$AP = BP \qquad (Given)$$

$$AQ = BQ \qquad (Given)$$

$$QP = QP \qquad (Common)$$

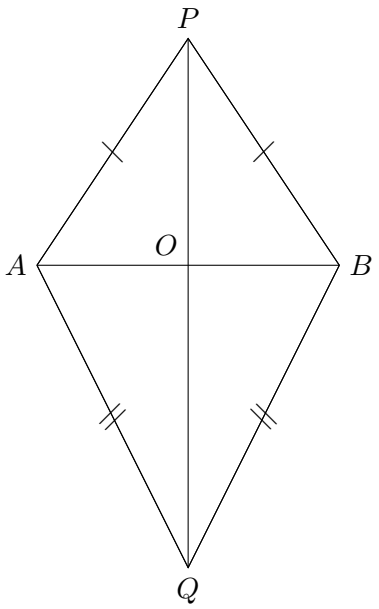



Figure 7.7.2

So,

$$\triangle APQ \cong \triangle BPQ \qquad (SSS\ rule)$$

Hence,

$$\angle APQ = BPQ \qquad (CPCT)$$

and

$$\angle AQP = \angle BQP \qquad (CPCT)$$

Now, in $\triangle PAO$ and $\triangle PBO$

$$\angle APO = \angle BPO \qquad (Proved)$$

$$AP = BP \qquad (Given)$$

$$PO = PO \qquad (Given)$$

So,

$$\triangle APO \cong \triangle BPO \qquad (SAS\ axiom)$$

Hence,

$$\angle AOP = \angle BOP \qquad (CPCT)$$

$$AO = BO \qquad (CPCT) \qquad (a)$$

$$\angle AOP + \angle BOP = 180^\circ \qquad (Linear\ pair\ angle)$$

$$2\angle AOP = 180^\circ \qquad (\angle AOP = \angle BOP)$$

Therefore,

$$\angle AOP + \angle BOP = 90^\circ \qquad (b)$$

So, with condition (a) and (b), PQ is perpendicular bisector of line-segment AB.

**Example 7.7.4.** P is a point equidistant from lines $OB$ and $OC$ intersecting at point O as in Fig. (7.7.3). Show that $OP$ bisects the angle between them.

**Solution:**

In Fig. (7.7.3), line-segment PB and PC perpendicular on line OB and OC respectively. So, $\triangle BOP$ and $\triangle COP$ are right angle triangles. Now in $\triangle BOP$ and $\triangle COP$,

$$PB = PC \qquad (Given)$$

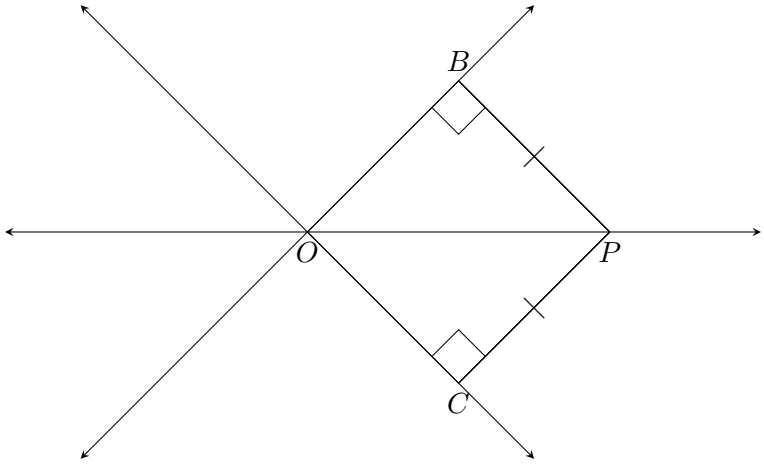



Figure 7.7.3

$$PO = PO \qquad (Common)$$

$$\angle POB = \angle POC \qquad (Given)$$

So,

$$\triangle BOP \cong \triangle COP \qquad (RHS\ rule)$$

Hence,

$$\angle POB = \angle POC \qquad (CPCT)$$

Therefore line OP is angle bisector of line OB and OC.

## 7.8 Solution of exercise 7.3

**Problem 7.8.1.** $\triangle ABC$ and $\triangle DBC$ are two isosceles triangle on the same base BC as in Fig. (7.8.1). If AD is extended to intersect BC at P, show that,

i $\triangle ABD \cong \triangle ACD$

ii $\triangle ABP \cong \triangle ACP$

iii AP bisects $\angle A$ as well as $\angle D$

iv AP is the perpendicular bisector of BC.

**Solution :** In Fig. (7.8.1), $\triangle ABC$ and $\triangle DBC$ are isosceles triangle. Take $AB = AC$ and $DB = DC$. In $\triangle ABD$ and $\triangle ACD$,

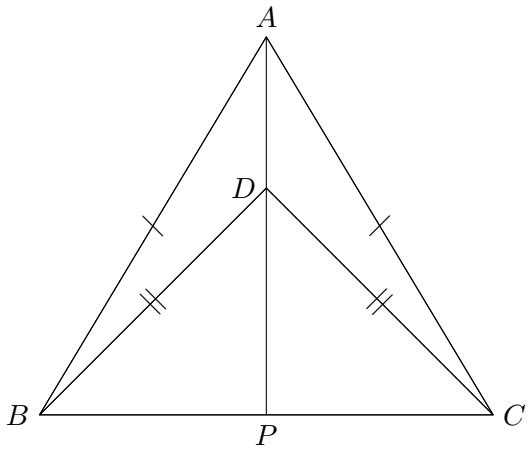



Figure 7.8.1

$$AB = AC \qquad (Given)$$

$$DB = DC \qquad (Given)$$

$$AD = AD \qquad (Common)$$

so,

$$\triangle ABD \cong \triangle ACD \qquad (SSS\ rule)$$

Hence,

$$\angle BAD = \angle CAD \qquad (CPCT)$$

Therefore, AD bisects $\angle A$, and

Now consider in $\triangle ABP$ and $\triangle ACP$,

$$\angle BAP = \angle CAP \qquad (Proved)$$

$$BA = CA \qquad (Given)$$

$$AP = AP \qquad (Common)$$

Hence,

$$\triangle BAP = \triangle CAP \qquad (SAS\ axiom)$$

So,

$$\angle APB = \angle APC \qquad (CPCT)$$

$$\angle APB = \angle APC = 90^\circ \qquad (Linear\ pair\ angle)$$

and

$$PB = PC \qquad (CPCT)$$

Therefore, AP is perpendicular bisectors of line segment BC.

**Problem 7.8.2.** AD is altitude of an isosceles triangle ABC in which $AB = AC$. Show that (i) AD bisects BC, (ii) AD bisects $\angle A$.

**Solution :** In Fig. (7.8.2), $\triangle ABC$ is an isosceles triangle with $AB = AC$. Let us take $\triangle ADB$ and $\triangle ADC$. AD is altitude of $\triangle ABC$. So, $AD \perp BC$. Hence $\triangle ADB$ and $\triangle ADC$ are right triangle.

$$AB = AC \qquad (Given)$$

$$AD = AD \qquad (Common)$$

$$\angle ADB = \angle ADC = 90^\circ \qquad (AD \perp BC)$$

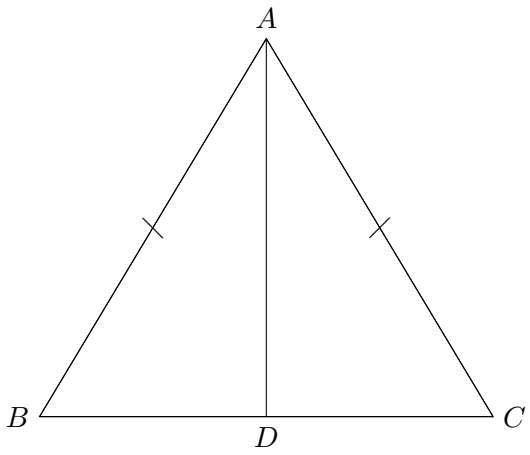



Figure 7.8.2

So,

$$\triangle ABD \cong \triangle ACD \qquad (RHS\ rule.)$$

Hence,

$$\triangle DAB = \triangle DAC \qquad (CPCT)$$

So, AD is angle bisector of $\angle A$. Now

$$DB = DC \qquad (CPCT)$$

So, AD is bisector of BC.

**Problem 7.8.3.** Two sides AB and BC and median AM of one $\triangle ABC$ are respectively equal to sides PQ and QR and median PN of $\triangle PQR$ as in Fig. (7.8.3). Show that :

i $\triangle AMB \cong \triangle PQN$

ii $\triangle ABC \cong \triangle PQR$

**Solution :** In Fig. (7.8.3), consider $\triangle ABM$ and $\triangle PQN$,

$$AB = PQ \qquad (Given)$$

$$AM = PN \qquad (Given)$$

$$MB = NQ \qquad (Given)$$

Hence

$$\triangle ABM \cong \triangle PQN \qquad (SSS\ rule)$$

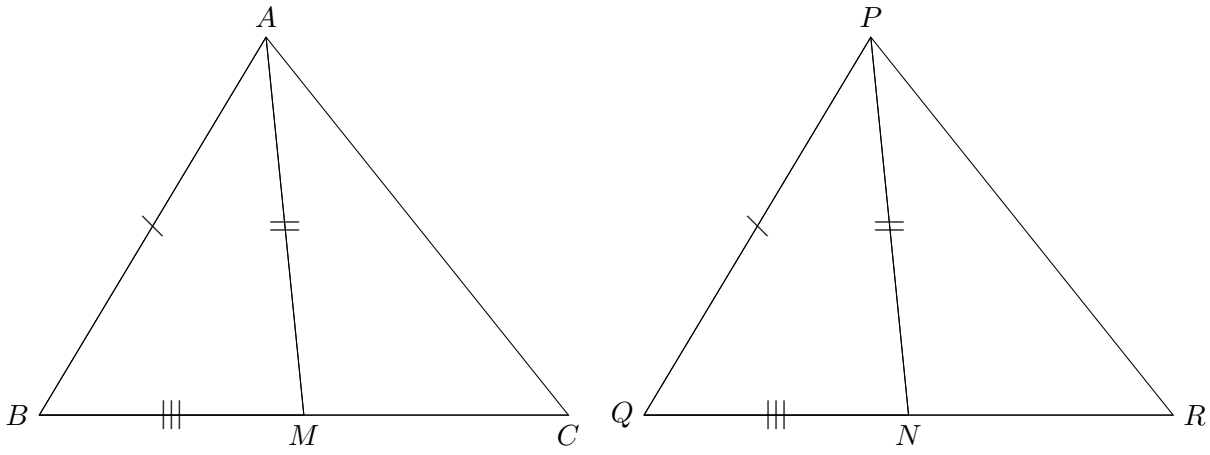



Figure 7.8.3

Now, in $\triangle ABC$ and $\triangle PQR$

$$AB = PQ \qquad (Given)$$

$$\angle ABM = \angle PQN \qquad (Proved)$$

$BC = QR$ $(AM \text{ and } PN \text{ are median of } \triangle ABC \text{ and } \triangle PQR \text{ respectively})$

$$\triangle ABC \cong \triangle PQR \qquad (SAS \text{ , axiom.})$$

**Problem 7.8.4.** BE and CF are two equal altitude of a $\triangle ABC$. Using RHS congruence rule. Prove that the $\triangle ABC$ is isosceles.

**Solution :** In Fig. (7.8.4), BE and CF are altitude of $\triangle ABC$ from vertex B and C respectively. So, $\triangle BFC$ and $\triangle CEB$ will be right angle triangle. Now in $\triangle BFC$ and $\triangle CEB$

$$CF = BE \qquad (Given)$$

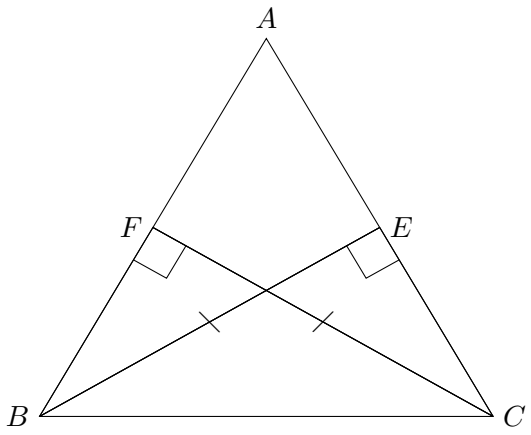



Figure 7.8.4

$$\angle CFB = \angle BEC = 90^\circ \qquad (Given)$$

$$CB = BC \qquad (Common\ hypotenuse)$$

Hence,

$$\triangle CFB \cong \triangle BEC \qquad (RHS)$$

So,

$$\angle FBC = \angle ECB\ or\ \angle ABC = \angle ACB \qquad (CPCT)$$

Therefore $\triangle ABC$ is an isosceles triangle.

**Problem 7.8.5.** $\triangle ABC$ is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.

**Solution :** In Fig. (7.8.5), $\triangle ABC$ is isosceles triangle with $AB = AC$. Now, in $\triangle ABP$ and $\triangle ACP$

$$AB = AC \qquad (Given)$$

$$AP = AP \qquad (Common)$$

$$\angle APB = \angle APC \qquad (Construction)$$

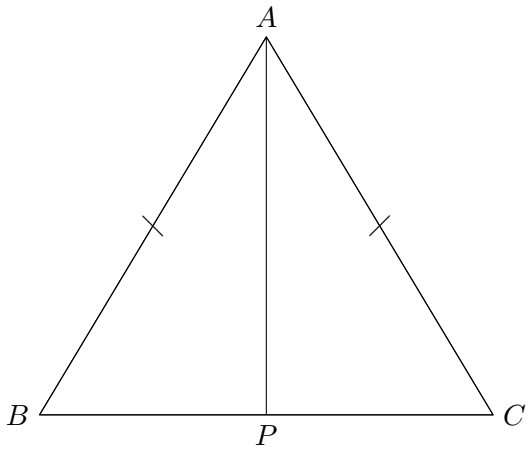



Figure 7.8.5

Hence,

$$\triangle ABP = \triangle ACP \qquad (RHS\ rule)$$

So,

$$\angle ABP = \angle ACP \qquad (CPCT)$$

$$\angle B = \angle C \qquad (CPCT)$$

# 8 QUADRILATERALS

## 8.1 Properties of a Parallelogram

**Quadrilateral :** A quadrilateral has **four sides, four angles and four vertices.**

**Parallelogram :** A parallelogram is a quadrilateral in which **both pair of opposite sides are parallel.**

**Theorem 8.1.1:** *A diagonal of a parallelogram divides it into two congruent triangles.*

*Proof.* Let ABCD be a parallelogram and AC be a diagonal and AC be a diagonal. In Fig. (8.1.1), the diagonal AC divides parallelogram into two triangles, namely, $\triangle ABC$ and $\triangle CDA$.

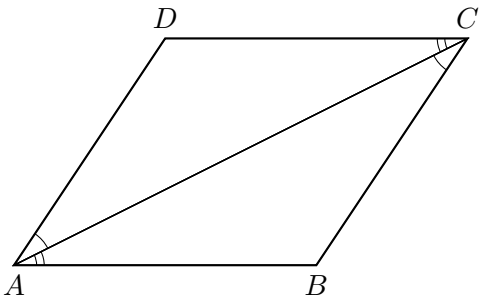



Figure 8.1.1

We need to prove that these triangles are congruent.

In $\triangle ABC$ and $\triangle CDA$, the $AB \parallel DC$ and AC is transversel line-segment. So,

$$\angle CAB = \angle ACD \qquad (pair\ of\ alternate\ angles)$$

Also, $BC \parallel AD$ and AC is transversal line-segment. So,

$$\angle ACB = \angle CAD \qquad (pair\ of\ alternate\ angles)$$

$$AC = AC \qquad (Common)$$

Hence,

$$\triangle ABC \cong \triangle CDA$$

therefore, diagonal AC divides parallelogram ABCD into two congruent $\triangle ABC$ and $\triangle CDA$. □

**Theorem 8.1.2:** *In a parallelogram, opposite sides are equal.*

*Proof.* In Fig. (8.1.2), ABCD is a parallelogram and AC is a diagonal.

From Theorem (8.1.1),

$$\triangle ABC \cong \triangle CDA$$

Hence,

$$BC = DA \qquad (CPCT)$$

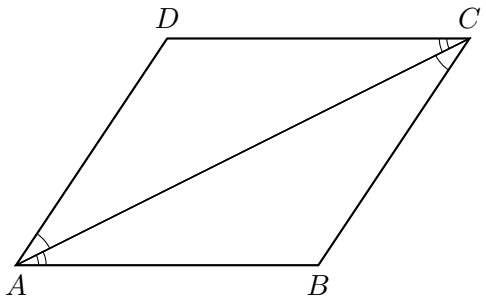



Figure 8.1.2

and

$$AB = DC \qquad (CPCT)$$

So, opposite sides in a parallelogram are equals. □

**Theorem 8.1.3:** *If aech pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

*Proof.* In Fig. (8.1.3), let sides $AB = CD$ in quadrilateral ABCD and also, $BC = AD$. Draw the diagonal AC.

AC is common in both triangle $\triangle ABC$ and $\triangle CDA$ Hence,

$$\triangle ABC \cong \triangle CDA$$

So,

$$\angle CAB = \angle ACD \qquad (CPCT)$$

$$\angle ACB = \angle CAD \qquad (CPCT)$$

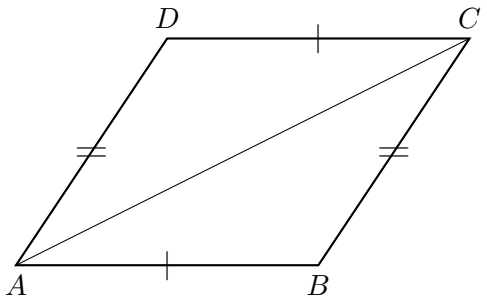



Figure 8.1.3

Therefore, $AB \parallel DC$ and $BC \parallel AD$. We concludes that ABCD is a parallelogram. □

**Theorem 8.1.4:** *In a parallelogram, opposite angle are equal.*

*Proof.* In Fig. (8.1.4), ABCD ia a parallelogram.

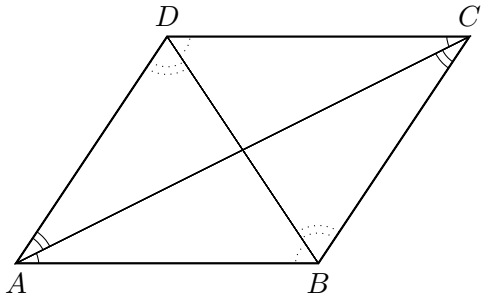



Figure 8.1.4

In parallelogram ABCD, AC is diagonal of quadrilateral ABCD. By The-

orem (8.1.1),

$$\triangle ABC \cong \triangle CDA$$

So,

$$\angle ABC = \angle CDA \qquad (CPCT)$$

Similarly,

$$\triangle ABD \cong \triangle CDB$$

So,

$$\angle BAD = \angle BCD \qquad (CPCT)$$

Hence, each pair of opposite angle in parallelogram are equals. □

**Theorem 8.1.5:** *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

*Proof.* In Fig. (8.1.5), ABCD is quadrilateral and diagonal AC and DB intersect each other at the point O. In a parallelogram the diagonal bisect to each other.

Now, in $\triangle AOD$ and $\triangle BOC$,

$$OA = OC \qquad (O \text{ is mid point } AC)$$

$$OB = OD \qquad (O \text{ is mid point } BD)$$

$$\angle DOA = \angle BOC \qquad (Verticaly\ opposite\ angle)$$

So,

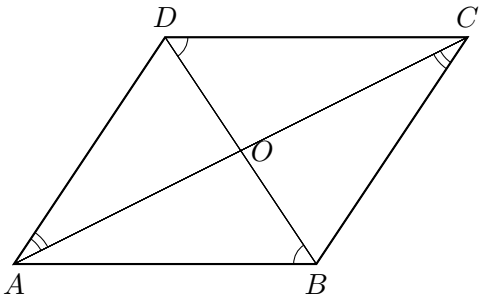



Figure 8.1.5

$$\angle DAO = \angle BCO \qquad (CPCT)$$

Ac is transversal line-segment intersect line-seg-ment AD and BC. So,

$$AD \parallel BC \qquad (Alternate\ pair\ angle\ are\ equal)$$

Similarly, we can prove,

$$AB \parallel DC \qquad (Alternate\ pair\ angle\ are\ equal)$$

Hence, ABCD is a parallelogram. □

**Theorem 8.1.6:** *The diagonal of a parallelogram bisect each other.*

*Proof.* This theorem can be prove by using parallelogram properties. □

**Theorem 8.1.7:** *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

*Proof.* It is converse of theorem (8.1.6). □

**Example 8.1.8.** Show that each angle of a rectangle is a right angle.

**Solution :** A rectangle is a parallelogram in which one angle is a right angle. Let ABCD be a rectangle in which $\angle A = 90°$. We have to show that $\angle A = \angle B = \angle C = \angle D = 90°$.

In Fig. (8.1.6), we have, $AD \parallel BC$ and $AB$ is a transversal line. So,

$\angle A + \angle B = 180°$ (*Interior angle on the same side of the transversal*)

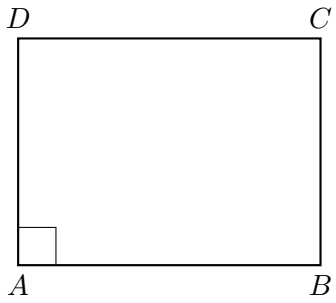



Figure 8.1.6

But, $\angle A = 90°$. So, $\angle B = 180 - \angle A = 90°$. Now,

$\angle C = \angle A$ *and* $\angle D = \angle B$ (*Opposite angle of the parallelogram* )

So,

$$\angle C = \angle D = 90^\circ.$$

Therefore, each of the angle of a rectangle is right angle.

**Example 8.1.9.** Show that the diagonal of a rhombus are perpendicular to each other.

**Solution :** In Fig.(8.1.7), consider the rhombus ABCD. We know that

$$AB = BC = CD = DA \qquad (Side\ of\ rhombus)$$

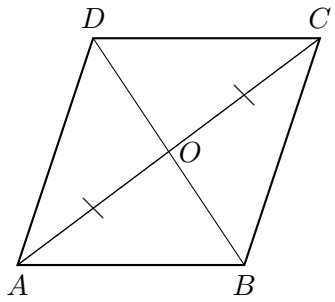



Figure 8.1.7

Now, in $\triangle AOD$ and $\triangle COD$,

$$OA = OC \qquad (Diagonal\ of\ a\ parallelogram\ bisect\ each\ other)$$

$$OD = OD \qquad (Comonn)$$

$$AD = CD \qquad (Side\ of\ rhombus.)$$

Therefore,

$$\triangle AOD \cong \triangle COD \qquad (SSS\ rule)$$

Hence,

$$\angle AOD = \angle COD \qquad (CPCT)$$

Here,

$$\angle AOD + \angle COD = 180^\circ \qquad (Linear\ pair)$$

So,

$$2\angle AOD = 180^\circ \implies \angle AOD = 90^\circ$$

So, the diagonal of a rhombus are perpendicular to each other.

**Example 8.1.10.** Two parallel line $l$ and $m$ are intersected by transversal $p$ as in Fig. (8.1.8). Show that the quadrilateral formed by the bisectors of interior angles is rectangle.

**Solution :** In Fig. (8.1.8), $PS \parallel QR$ and transversal $p$ intersects them at points A and C respectively.

The bisector of $\angle PAC$ and $\angle ACQ$ intersect at B and bisectors of $\angle ACR$ and $\angle SAC$ intersect at D.

We need to show that quadrilateral ABCD is a rectangle.

Now,

$$\angle PAC = \angle ACR \qquad (Alternate\ angle\ pair)$$

So,

$$\frac{1}{2}\angle PAC = \frac{1}{2}\angle ACR$$

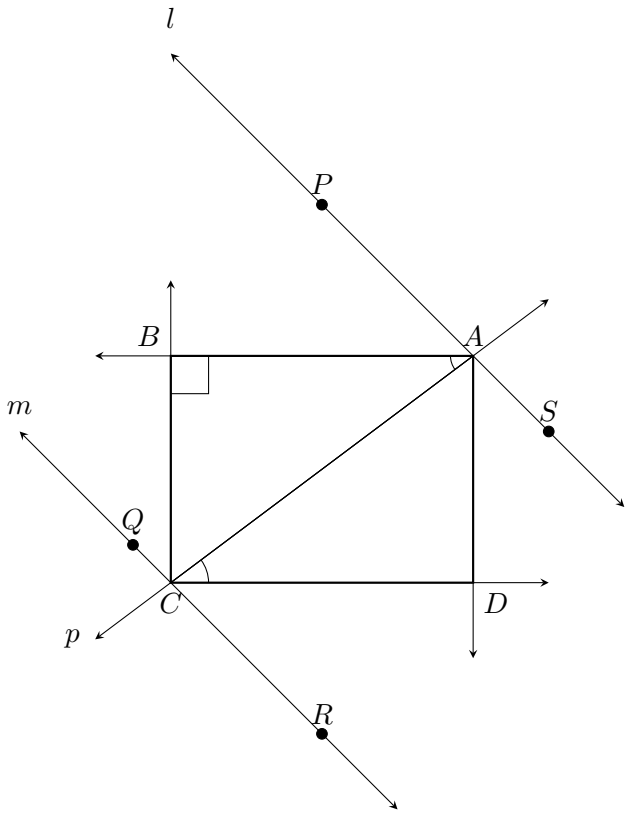



Figure 8.1.8

$$\angle BAC = \angle ACD$$

These form a pair of alternate angle for lines AB and CD with AC as transver-

sal and they are equal also. So,

$$AB \parallel DC$$

Similarly,

$$BC \parallel AD \qquad (Considering\ \angle ACB\ and\ \angle CAD\ )$$

Therefore, quadrilateral ABCD is a parallelogram. So,

$$\angle PAC + \angle CAS = 180^\circ \qquad (Linear\ pair)$$

Hence,

$$\frac{1}{2}\angle PAC + \frac{1}{2}\angle ACR = \frac{1}{2} \times 180^\circ = 90^\circ$$

$$\implies \angle BAC + \angle CAD = 90^\circ$$

$$\implies \angle BAD = 90^\circ$$

So, ABCD is a parallelogram in which one angle is $90^\circ$. Therefore ABCD is a rectangle.

**Example 8.1.11.** ABC is an isosceles triangle in which $AB = AC$. AD bisects exterior $\angle PAC$ and $CD \parallel AB$ as in Fig. (8.1.9). Show that (i) $\angle DAC = \angle BCA$ and (ii) ABCD is a parallelogram.

**Solution :** (i) $\triangle ABC$ is an isosceles in which $AB = AC$

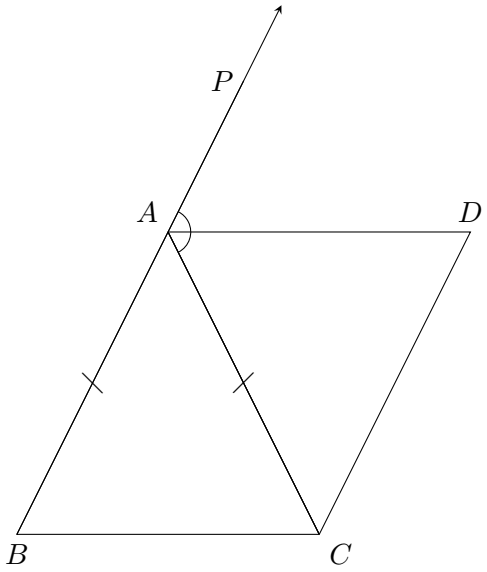



Figure 8.1.9

So,

$$\angle ABC = \angle ACB \qquad (Angles\ opposite\ equal\ sides)$$

Also,

$$\angle PAC = \angle ABC + \angle ACB \qquad (Exterior\ angle\ of\ a\ triangle)$$

$$\angle PAC = 2\angle ACB \tag{8.1.1}$$

Now, AD bisects $\angle PAC$.

$$\angle PAC = 2\angle DAC \tag{8.1.2}$$

Therefore,

$$2\angle DAC = 2\angle ACB \qquad [From\ (8.1.1)\ and\ (8.1.2)]$$

Hence,

$$\angle DAC = \angle ACB$$

(ii) Now, these two angles form a pair of alternate angles when transversal line intersected line segment AD and BC.

So,

$$BC \parallel AD$$

Also,

$$BA \parallel CD$$

Now, both pairs of opposite side of quadrilateral ABCD are parallel. So ABCD is a parallelogram.

## 8.2 Solution of Exercise 8.1

**Problem 8.2.1.** If the diagonal of a parallelogram are equal, then show that it is a rectangle.

**Solution :** ABCD is a parrallelogram with diagonal $AC = BD$.

See the Fig. (8.2.1), in $\triangle ABC$ and $\triangle BCD$

$$BC = BC \qquad (Common)$$

$$BA = DC \qquad (Opposite\ side\ of\ parallelogram)$$

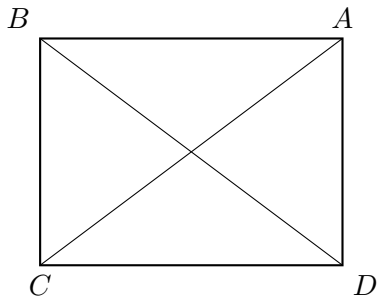



Figure 8.2.1

$$AC = BD \qquad (Given)$$

Hence

$$\triangle ABC \cong \triangle BCD \qquad (SSS\ rule)$$

So,

$$\angle ABC = \angle BCD \qquad (CPCT)$$

$$\angle A + \angle B + \angle C + \angle D = 360^\circ \qquad (All\ angle\ of\ parallelogram)$$

$$2\angle B + 2\angle C = 360^\circ \qquad (Parallelogram\ properties)$$

$$\angle B + \angle C = 180^\circ \qquad (Parallelogram\ property)$$

$$\angle B = 90^\circ \qquad (\angle B = \angle C)$$

Therefore, ABCD is a rectangle.

**Problem 8.2.2.** Show that the diagonals of a square are equal and bisect each other at right angles.

**Solution :** In Fig. (8.2.2), ABCD is a squre with diagonal AC and BD.

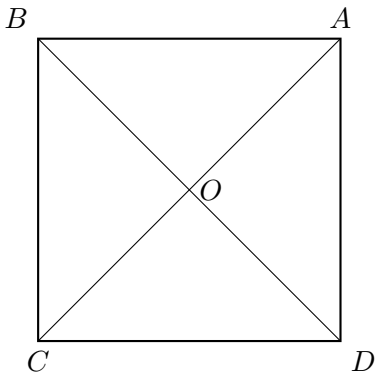



Figure 8.2.2

In $\triangle ABC$ and $\triangle BCD$

$$\angle B = \angle C \qquad (Angle\ of\ square)$$

$$BC = BC \qquad (Common)$$

$$AB = CD \qquad (Side\ of\ square)$$

So,

$$\triangle ABC \cong \triangle BCD \qquad (SAS\ rule)$$

Hence,

$$AC = BD \qquad (CPCT)$$

Hence, diagonal of squre are equals. Let us take $\triangle BOC$ and $\triangle AOD$

$$\angle OAD = \angle OCB \qquad (AD \parallel BC)$$

$$\angle ODA = \angle OBC \qquad (AD \parallel BC)$$

$$AD = CB \qquad (Side\ of\ squre)$$

So,

$$OD = OB \qquad (CPCT)$$

$$OA = OC \qquad (CPCT)$$

Therefore, diagonal of squre bisect each other.

**Problem 8.2.3.** In Fig. (8.2.3), diagonal AC of parallelogram ABCD bisect $\angle A$. Show that,

i it bisect $\angle C$ also,

ii ABCD is a rhombus.

**Solution :** ABCD is a parallelogram.

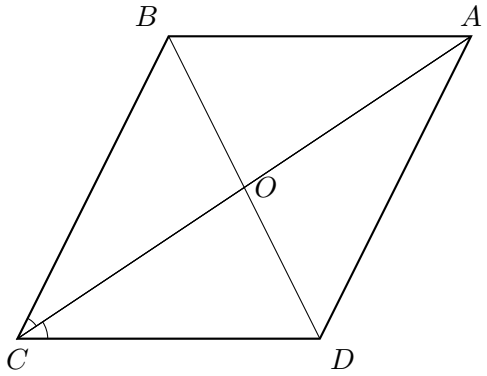



Figure 8.2.3

In parallelogram ABCD

$$\angle C = \angle A \qquad (Opposite\ angle\ of\ parallelogram)$$

$$\angle BCA = \angle ACD \qquad (Given) \quad (i)$$

$$\angle BCA = \angle CAD \qquad (BC \parallel AD)$$

$$\angle ACD = \angle CAB \qquad (BA \parallel CD)$$

Therefore,

$$\angle CAB = \angle CAD \qquad (By\ i)$$

So, AC bisect $\angle C$. Diagonal of parallelogram ABCD bisect each other, so in $\triangle ABC$, OB will be angle bisector of $\angle B$, similarly OD will be angle bisector of $\angle D$. So,

$$\frac{1}{2}\angle B = \frac{1}{2}\angle D \implies \angle OBC = \angle ODC$$

Now, in $\triangle BCO$ and $\triangle DCO$

$$OC = OC \qquad (Common)$$

$$\angle BOC = \angle DOC \qquad (Given)$$

$$\angle OBC = \angle ODC \qquad (Proved)$$

Hence

$$\triangle BCO \cong \triangle DCO \qquad (AAS\ rule)$$

$$BC = CD \qquad (CPCT)$$

Similarly,

$$AB = BC = CD = DA$$

Therefore, ABCD is a rhombus.

**Problem 8.2.4.** ABCD is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that : (i) ABCD is a square, (ii) diangole BD bisects $\angle B$ as well as $\angle D$.

**Solution :** In Fig. (8.2.4), ABCD is a rectangle. Therefore, ABCD is a parallelogram also.

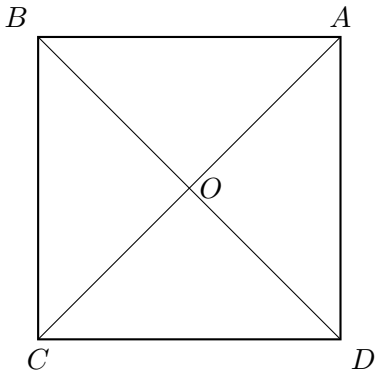



Figure 8.2.4

The diagonal of parallelogram divides two congruent triangle to parallelogram. So,

$$\triangle ABC \cong \triangle ADC$$

$$\angle ACB = \angle ACD \qquad (Given)$$

$$\angle BAC = \angle DAC \qquad (Given)$$

Hence,

$$CB = CD \qquad (CPCT)$$

$$AB = AD \qquad (CPCT)$$

Hence, ABCD is square. Similarly, BD also, bisect parallelogram ABCD into two congruent triangle. So,

$$\triangle ABD \cong \triangle BCD \qquad (Parallelogram\ properties)$$

Hence,

$$\angle ABD = \angle DBC \qquad (CPCT)$$

$$\angle ADB = \angle BDC \qquad (CPCT)$$

Therefore, BD bisect $\angle B$ and $\angle D$.

**Problem 8.2.5.** In parallelogram ABCD, two points P and Q are taken on diagonal BD such that $DP = BQ$ as in Fig.(8.2.5). Show that :

i $\triangle APD \cong \triangle CQB$

ii $AP = CQ$

iii $\triangle AQB \cong \triangle CPD$

iv $AQ = CP$

v APCQ ia a parallelogram

**Solution :** ABCD is a parallelogram and BD is diagonal. Hence,

$$\triangle ABD \cong triangleBDC$$

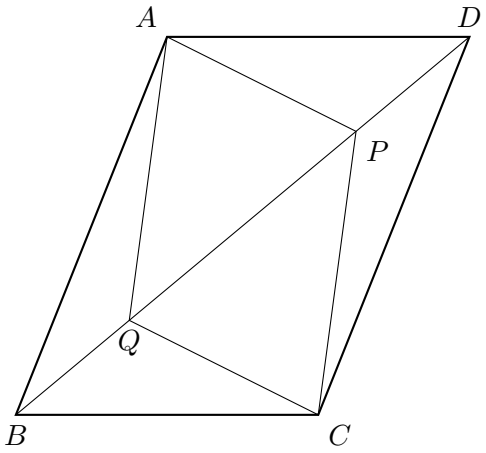



Figure 8.2.5

So,

$$\angle ADB = \angle DBC \qquad (CPCT)$$

$$AD = BC \qquad (CPCT)$$

and

$$DP = BQ \qquad (Given)$$

Therefore,

$$\triangle APD \cong \triangle CQB \qquad (SAS\ rule)$$

so,

$$AP = CQ \qquad (CPCT)$$

Now, in $\triangle ABQ$ and $\triangle CDP$

$$AB = DC \qquad (Side\ of\ parallelogram)$$

$$BQ = DP \qquad (Given)$$

$$\angle QBA = \angle PDA \qquad (Parallelogram\ properties)$$

Hence,

$$\triangle ABQ \cong \triangle CDP \qquad (SAS\ rule)$$

So,

$$AQ = CP \qquad (CPCT)$$

From, $\triangle ABQ \cong \triangle CDP$ and $\triangle APD \cong \triangle CQB$ So, $AP = CQ$ and $AQ = CP$, Therefore, APCQ is a parallelogram.

**Problem 8.2.6.** ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD as in Fig. (8.2.6). Show that :

i $\triangle APB \cong \triangle CQD$

ii AP =CQ

**Solution :** ABCD is a parallelogram.

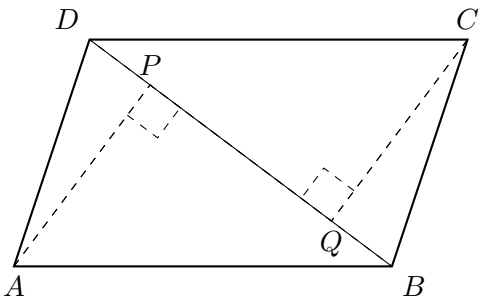



Figure 8.2.6

The $\triangle APQ$ and $\triangle CQD$ has common base DB, which is diagonal of parallelogram ABCD. Therefore,

$$AP = CQ \qquad (Heights\ of\ congruent\ triangle)$$

Now in $\triangle APB$ and $\triangle CQD$, $\triangle APB$ and $\triangle CQD$ are right angle triangle with AP and CQ heights respevtively.

$$AP = CQ \qquad (Opposite\ side\ of\ parallelogram)$$

$$AP = CQ \qquad (Proved)$$

So,

$$\triangle APB \cong \triangle CQD \qquad (RHS\ rule)$$

**Problem 8.2.7.** ABCD is a trapezium in which $AB \parallel CD$ and $AD = BC$ as in Fig. (8.2.7). Show that :

i $\angle A = \angle B$

ii $\angle C = \angle D$

iii $\triangle ABC = \triangle BAD$

iv *diagonal* $AC =$ *diagonal* $BD$

**Solution :** ABCD is a trapezium. Extended AB both side and draw line through C and D parallel AD and BC intersecting AB produced at E and F respectively.

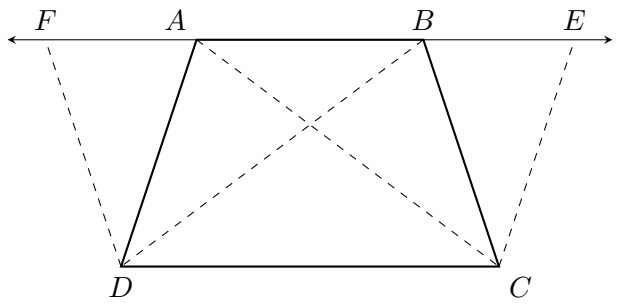



Figure 8.2.7

Now, $\triangle BCE \cong \triangle ADF$ $(i)$ by construction.

The parallelogram AECD and BFDC are congruent parallelogram, so triangle

produced by diagonal of parallelogram will be congruent. Hence,

$$\triangle BDF \cong \triangle ACE \qquad (ii)$$

From (i) and (ii) you get,

$$\triangle ABC \cong \triangle BAD$$

$$\angle A = \angle B \qquad (CPCT)$$

Similarly,

$$\angle C = \angle D$$

$$AC = BD \qquad (CPCT)$$

Therefore, *diagonal* $AC$ = *diagonal* $BD$.

## 8.3 The Mid-point Theorem

**Theorem 8.3.1:** *The line segment joining the mid-points of a triangle is parallel to the third side.*

*Proof.* Let $\triangle ABC$ has mid-points E and F of AB and AC respectively. Draw line CD parallel with BA intersect EF produced D.

In Fig.(8.3.1), E and F are mid-points of AB and AC respectively and $CD \parallel BA$.

In $\triangle AEF$ and $\triangle FCD$

$$\angle EAF = \angle FCD \qquad (BA \parallel CD)$$

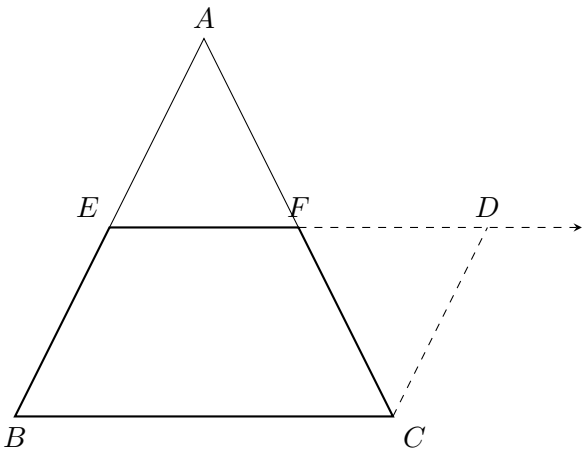



Figure 8.3.1

$$\angle AFE = \angle DFC \qquad (Vertically\ opposite\ angle)$$

$$AF = FC \qquad (Given)$$

Hence,

$$\triangle EAF = \angle FCD \qquad (SAS\ axiom)$$

So,

$$EF = FD \qquad (CPCT)$$

$$BE = EA = CD \qquad (CPCT)$$

Therefore, BCDE is a parallelogram. Hence, $EF \parallel BC$.

$$EF = \frac{1}{2}BC \qquad (parallelogram\ proprty)$$

□

**Theorem 8.3.2:** *The line drawn through the mid-point of one side of a triangle parallel to another side, bisects the third side.*

*Proof.* In Fig. (8.3.2),

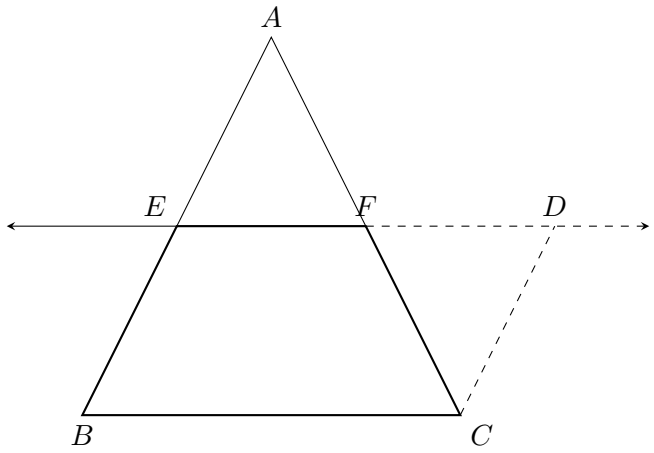



Figure 8.3.2

In Fig.(8.3.2), E and F are mid-points of AB and AC respectively and $CD \parallel BA$. Consider triangle $\triangle AEF$ and $\triangle CDF$

$$AF = \angle CFD \qquad (F\ is\ mid-point\ of\ AC)$$

$$\angle EAF = \angle FCD \qquad (BA \parallel CD)$$

$$\angle AEF = \angle CFD \qquad (Vertically\ opposite\ angle)$$

Therefore,

$$\triangle AEF \cong \triangle CFD \qquad (ASA\ rule)$$

Hence,

$$EF = DF \text{ and } BE = AE = DC \qquad (CPCT)$$

Hence, ABCD is a parallelogram. So,

$$EF \parallel BC.$$

□

**Example 8.3.3.** In $\triangle ABC$ D, E and F are respectively the mid-points of the sides AB, BC and CA as in Fig.(8.3.3). Show that $\triangle ABC$ is divided into four congruent triangle by joining D, E and F.

**Solution :** In $\triangle ABC$ D and E are mid-points of sides AB and BC. Therefore, according theorem (8.3.2),

$$DE \parallel AC \qquad By\ Theorem(8.3.2)$$

Similarly,

$$EE \parallel BA \qquad By\ Theorem(8.3.2)$$

Therefore, ADEF, BDFE and DFCE are parallelograms. Now DE is a diagonal of the parallelogram BDFE, therefore,

$$\triangle BDE \cong \triangle FED \qquad (Parallelogram\ properties)$$

$$\triangle DAF \cong \triangle FED \qquad (Parallelogram\ properties)$$

$$\triangle EFF \cong \triangle FED \qquad (Parallelogram\ properties)$$

So, all the four triangles are congruent.

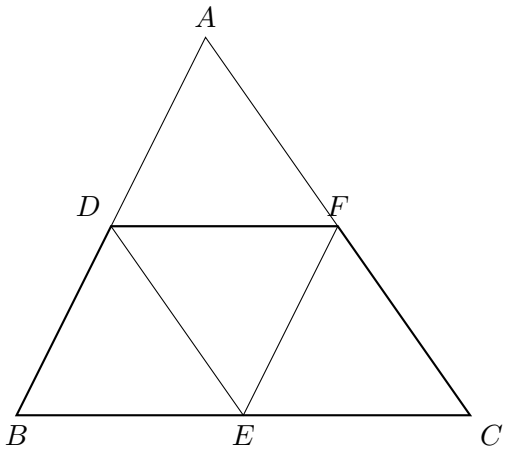



Figure 8.3.3

**Example 8.3.4.** In Fig.(8.3.4), $l, m$ and $n$ are three parallel lines intersected by transversals $p$ and $q$ such that $l, m$ and $n$ cut off equal intercepts AB and BC on $p$. Show that $l, m$ and $n$ cut off equal intercepts DE and EF on $q$ also.

**Solution :** In Fig.(8.3.3), line $l, m$ and $n$ are parallel and intercepts equal cot off AB and BC.

In $\triangle ACF$, the line $m$ bisects AC and parallel to third side CF. Therefore,

by Thereom (8.3.3),

$$AG = GF$$

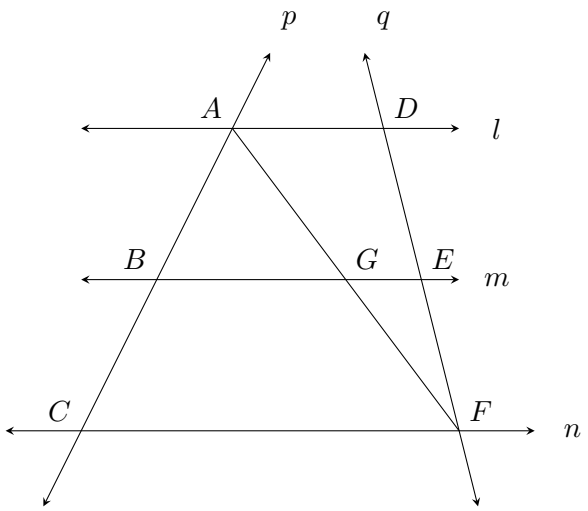



Figure 8.3.4

Now, in $\triangle AFD$, line $m$ bisects side AF and parallel to third side AD. Therefore,

$$DE = EF \qquad (Theorem\ 8.3.3)$$

## 8.4 Solution of Exercise 8.2

**Problem 8.4.1.** ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA as in Fig. (8.4.1). AC is diagonal. Show that:

i $SR \parallel AC$ and $SR = \frac{1}{2}AC$

ii $PQ = SR$

iii PQRS is a parallelogram

**Solution :** In Fig. (8,4.1), ABCD is a quadrilateral and P, Q, R and S are mid-points of sides of quadrilateral.

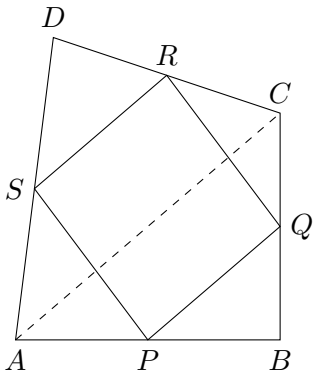



Figure 8.4.1

In $\triangle ADC$, S and R are mid-points of sides AD and BC. Therefore, by Theorem (8.3.1),

$$SR \parallel AC \qquad (by\ mid-points\ theorem)$$

Similarly,

$$PQ \parallel AC \qquad (mid-point\ theorem)$$

So,

$$SR \parallel PQ \qquad (SR\ and PQ\ \parallel AC)$$

and

$$SR = \frac{1}{2}AC \qquad (mid - point\ theorem)$$

Similarly,

$$PQ = \frac{1}{2}AC \qquad (mid - point\ theorem)$$

So,

$$SR = PQ$$

Similarly, we can show,

$$SP \parallel RQ\ and\ SP = RQ$$

Hence, PQRS is a parallelogram.

**Problem 8.4.2.** ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.

**Solution :** In Fig. (8.4.2), ABCD is a rhombus, P, Q, R ans S are mid-points of the sides AB, BC, CA and DA respectively. Consider in $\triangle ADC$, S and R are mid-points of sides AD and DC. So,

(8.4.1) $$SR \parallel AC\ and\ SR = \frac{1}{2}AC \qquad (Theorem\ 8.3.1)$$

Similarly,

(8.4.2) $$PQ \parallel AC\ and\ PQ = \frac{1}{2}AC \qquad (Theorem\ 8.3.1)$$

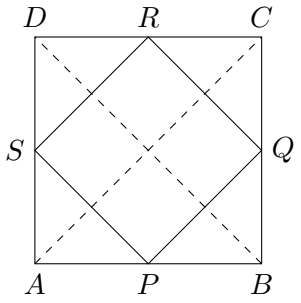



Figure 8.4.2

In $\triangle DBC$ R and Q are mid-points of sides DC and BC respectively. So,

(8.4.3) $$RQ \parallel DB \text{ and } RQ = \frac{1}{2}DB \qquad (Theorem\ 8.3.1)$$

Similarly,

(8.4.4) $$SP \parallel DB \text{ and } SP = \frac{1}{2}DB \qquad (Theorem\ 8.3.1)$$

(8.4.5) $$AC \perp BD \qquad (Rhombus\ properties)$$

From equation (8.4.1) and (8.4.3) and (8.4.5)

$$SR \perp RQ$$

and from (8.4.1), (8.4.2) and (8.4.3), (8.4.4)

$$SR = PQ \text{ and } SR \parallel PQ$$

$$SP = RQ \text{ and } SP \parallel RQ$$

Therefore, PQRS is a rectangle.

**Problem 8.4.3.** ABCD is a rectangle and P, Q, R and S are mid-points AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.

**Solution :** In Fig. (8.4.3), ABCD is a rectangle, P, Q, R ans S are mid-points of the sides AB, BC, CA and DA respectively. Consider in $\triangle ADC$, S and R are mid-points of sides AD and DC. So,

$$SR \parallel AC \text{ and } SR = \frac{1}{2}AC \qquad (Theorem\ 8.3.1) \tag{8.4.6}$$

Similarly,

$$PQ \parallel AC \text{ and } PQ = \frac{1}{2}AC \qquad (Theorem\ 8.3.1) \tag{8.4.7}$$

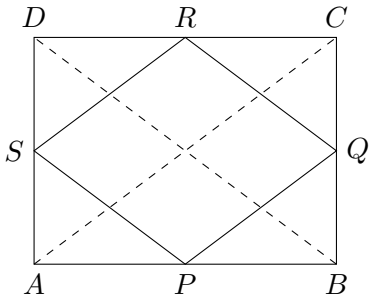



Figure 8.4.3

In $\triangle DBC$ R and Q are mid-points of sides DC and BC respectively. So,

(8.4.8) $RQ \parallel DB \text{ and } RQ = \frac{1}{2}DB$ (*Theorem* 8.3.1)

Similarly,

(8.4.9) $SP \parallel DB \text{ and } SP = \frac{1}{2}DB$ (*Theorem* 8.3.1)

(8.4.10) $AC = BD$ (*Diagonal of rectangle*)

From equation (8.4.6), (8.4.7), (8.4.8), (8.4.9) and (8.4.10),

$$PQ = QR = RS = SP$$

Therefore, PQRS is a hrombus.

**Problem 8.4.4.** ABCD is a trapezium in which $AB \parallel DC$, BD is a diagonal and E is mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F as in Fig. (8.4.4). Show that F is mid-point of BC.

**Solution :** In Fig. (8.4.4), ABCD is a trapezium in which $AB \parallel DC$ and E is mid-point of side AD. Now in $\triangle ABD$, E is mid-point of side AD.

$$EG \parallel AB \quad (Construction)$$

Therefore,

$$DG = GB \quad (Theorem\ 8.3.2)$$

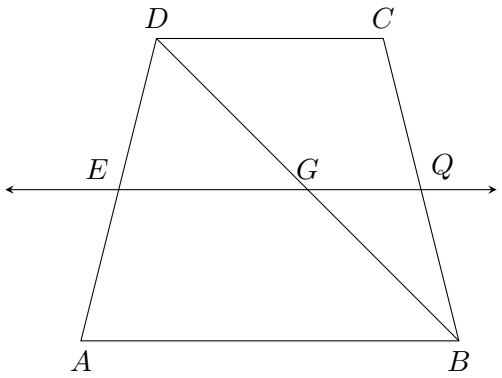



Figure 8.4.4

Now in $\triangle BDC$, G is mid-point of side DB.

$$GQ \parallel DC \qquad (Construction)$$

Therefore, Q is mid-point of side BC. Hence, EQ bisects BC.

**Problem 8.4.5.** In parallelogram ABCD, E and F are the mid-points of sides AB and DC respectively as in Fig. (8.4.5). Show that the line segment AF and EC trisect the diagonal BD.

**Solution :** In Fig. (8.4.5), ABCD is a parallelogram, E and F are mid-points of sides AB and DC respectively. Line segment $AB \parallel DC$ *and* $AD \parallel BC$. The transversal line-segment AF and EC intersect equal line-segment AE and FC on AB and DC respectively. Therefore,

$$AF \parallel EC$$

Now in $\triangle DQC$, F is mid-point of side DC and line-segment $FP \parallel QC$. So, P bisects line-segment DQ.

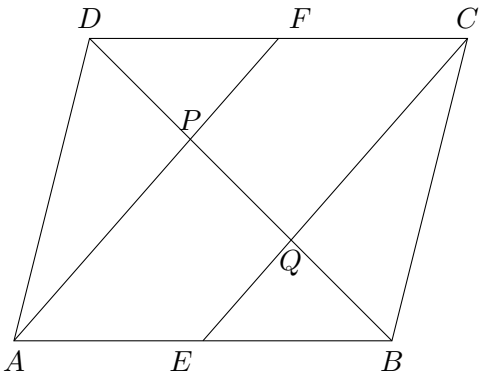



Figure 8.4.5

(8.4.11) $DP = PQ$ (*By theorem* 8.3.2)

Similarly, in $\triangle APB$, we can prove that,

(8.4.12) $PQ = QB$ (*By theorem* 8.3.2)

Therefore from equation (8.4.11) and (8.4.12)

$$DP = PQ = QB$$

Hence, line-segment AF and EC trisect the diagonal DB.

**Problem 8.4.6.** ABC is a triangle right angle at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that

i D is the mid-point of AC

ii $MD \perp AC$

iii $CM = MA = \frac{1}{2}AB$

**Solution :**

In Fig. (8.4.6), $\triangle ABC$ is right angle triangle with right angle at C. M is mid-point of side AB and line $MD \parallel BC$.

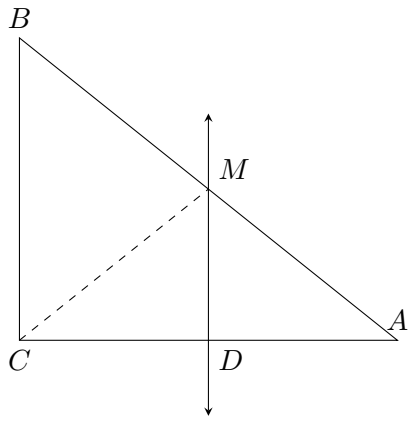



Figure 8.4.6

Therefore, MD bisects line-segment CA at D. So,

$$CD = DA \qquad (Theorem\ 8.3.2)$$

Hence, D is mid-point of AC.

$$MD \parallel BC \qquad (Construction)$$

So,

$$MD \perp AC \qquad (MD \parallel BC)$$

In $\triangle AMC$, perpendicular line-segment MD from vertex M, bisect base AC, So, $\triangle AMC$ is isosceles. Hence,

$$CM = AM = \frac{1}{2}AB.$$

## 8.5 Brain Teaser Question

**Problem 8.5.1.** In Fig. (8.5.1), ABCD is a parallelogram. Find $x$.

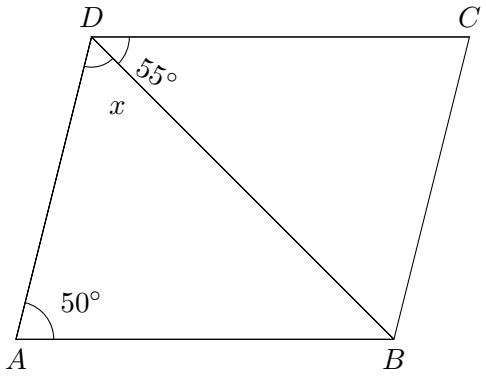



Figure 8.5.1

**Problem 8.5.2.** In Fig. (8.5.2), $PQ \parallel SR$ and $PS = QR$. Find $2x + y$.

**Problem 8.5.3.** ABCD is a parallelogram as in Fig. (8.5.3). Find the value of $x$ and $y$.

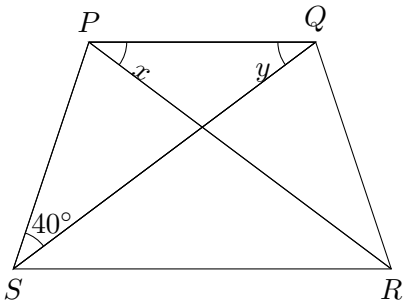



Figure 8.5.2

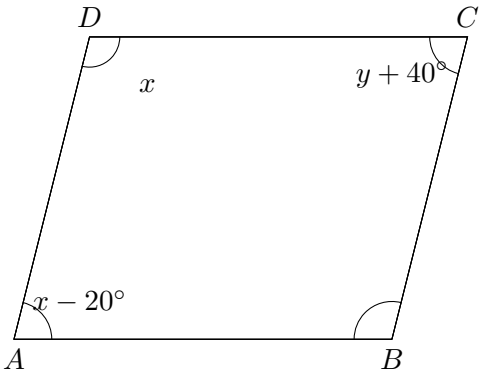



Figure 8.5.3

**Problem 8.5.4.** In Fig. (8.5.4), side $SR = RQ$ and $SP = QP$. Find the value of $x$ and $y$.

**Problem 8.5.5.** In Fig. (8.5.5), ABCD is an quadrilateral. The side AB and CD are produced F and E respectively. Find the value $x$.

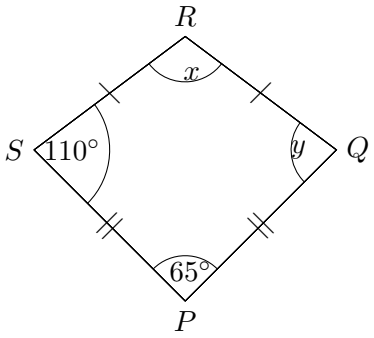



Figure 8.5.4

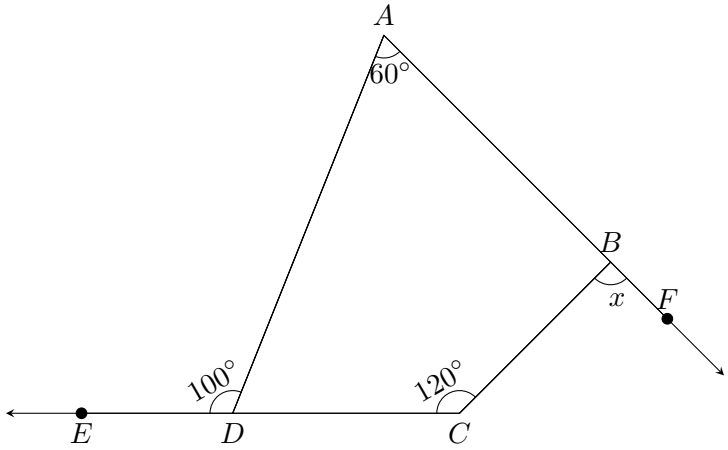



Figure 8.5.5

# 9 CIRCLE

## 9.1 Introduction

In the this chapter you will study about circle and their parts, chord and angle substended by a chord at a point, their distance from center, distance between two chords, angle substended by an arc of a circle, cyclic quadrilateral.

## 9.2 Angle Substended by a Chord at a point

**Definition 9.2.1.** Let a point R which is not contained in line-segment PQ. Join PR and QR. Then $\angle PRQ$ is called the **angle subtended by the line segment** PQ at R.

**Definition 9.2.2.** Let PQ is a chord and O is center of circle then $\angle POQ$ is called **angle substended by chord** PQ at the center O.

**Definition 9.2.3.** Let PQ is a chord of a circle with center O then arc PRQ and arc PSQ are called **major** and **minor** arcs PQ.

**Theorem 9.2.4:** *Equal chords of a circle subtend equal angles at the center.*

*Proof.* We have two equal chords AB and CD of a circle with centre O as in Fig. (9.2.1). We need to prove $\angle AOB = \angle COD$

In $\triangle AOB$ and $\triangle COD$,

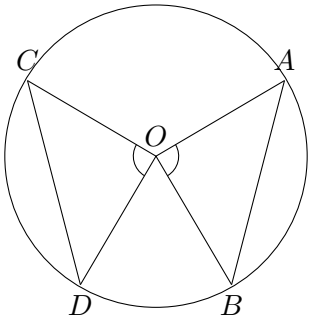



Figure 9.2.1

$$OA = OC \qquad (Radius\ of\ circle)$$

$$OB = OD \qquad (Radius\ of\ circle)$$

$$AB = CD \qquad (Given)$$

Therefore,

$$\triangle AOB \cong \triangle COD \qquad (SSS\ rule)$$

Hence,

$$\angle AOB = \angle COD \qquad (CPCT)$$

□

**Theorem 9.2.5:** *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.*

*Proof.* We have two equal chords AB and CD of a circle with centre O as in Fig. (9.2.2). We need to prove $\angle AOB = \angle COD$

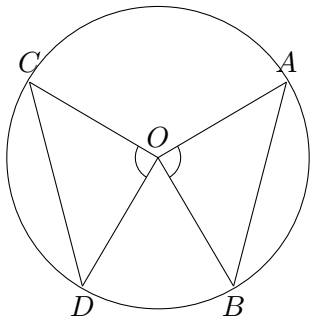



Figure 9.2.2

In $\triangle AOB$ and $\triangle COD$,

$$OA = OC \qquad (Radius\ of\ circle)$$

$$OB = OD \qquad (Radius\ of\ circle)$$

$$\angle AOB = \angle COD \qquad (Given)$$

Therefore,

$$\triangle AOB \cong \triangle COD \qquad (SAS\ axiom)$$

Hence,

$$AB = CD \qquad (CPCT)$$

□

## 9.3 Solution of Exercise 9.1

**Problem 9.3.1.** Recall that two circle are congruent if they have the same radii. Prove that equal chords of congruent circle subtend equal angles at centres.

**Solution :** Let there are two congruent circle with centre O and O' with radius $2cm$. The chord AB and CD are equal as in Fig. (9.3.1).

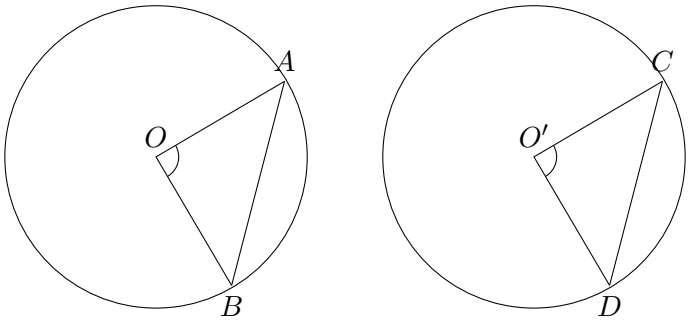



Figure 9.3.1

Now in $\triangle AOB$ and $\triangle CO'D$

$$OA = O'C \qquad (Radii\ of\ congruent\ circle)$$

$$OB = O'D \qquad (Radii\ of\ congruent\ circle)$$

$$AB = CD \qquad (Given)$$

Hence,

$$\triangle AOB \cong \triangle CO'D \qquad (SSS\ rule)$$

So,

$$\angle AOB = \angle CO'D \qquad (CPCT)$$

Therefore, equal chord of congruent cicrles substend equal angle at the centres.

**Problem 9.3.2.** Prove that if chords of congruent circles subtend equal angles at their centres, then the chords are equal.

**Solution :** Let there are two congruent circle with centre O and O' with radius $2cm$. The chord AB and CD are equal as in Fig. (9,3.2).

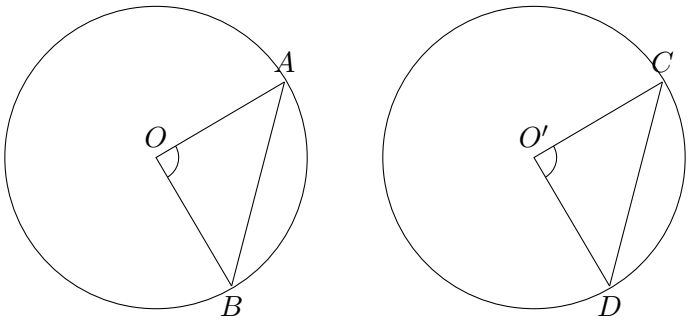



Figure 9.3.2

Now in $\triangle AOB$ and $\triangle CO'D$

$$OA = O'C \qquad (Radii\ of\ congruent\ circle)$$

$$OB = O'D \qquad (Radii\ of\ congruent\ circle)$$

$$\angle AOB = \angle COD \qquad (Given)$$

Hence,

$$\triangle AOB \cong \triangle CO'D \qquad (SAS\ axiom)$$

So,

$$AB = CD \qquad (CPCT)$$

Therefore, if chord of congruent cicrles substend equal angle at the centres, then chords will be equal.

## 9.4 Perpendicular from the Centre to a Chord and Distance of Equal Chords from the Centre

**Theorem 9.4.1:** *The perpendicular from the centre of a circle to a chord bisects the chord.*

*Proof.* In Fig. (9.4.1), AB is a chord of a circle with centre O. The line-segment $OM \perp AB$ and draw the line-segment OA and OB.

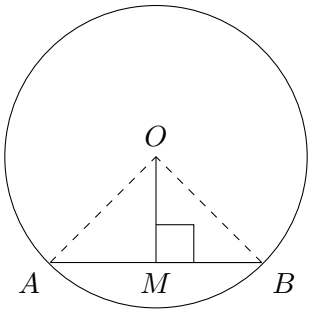



Figure 9.4.1

The $\triangle AMO$ and $\triangle BMO$ are right angle triangle with right angle $\angle AMO$ and $\angle BMO$.

In $\triangle AMO$ and $\triangle BMO$

$$OA = OB \qquad (Radii\ of\ circle)$$

$$OM = OM \qquad (Common\ side)$$

So,

$$\triangle AMO \cong \triangle BMO \qquad (RHS\ Rule)$$

Hence,

$$AM = BM \qquad (CPCT)$$

Therefore, perpendicular line-segment drawn from centre to the chord AB bisects the chord AB. □

**Theorem 9.4.2:** *The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.*

*Proof.* In Fig. (9.4.2), AB is a chord of a circle with centre O. The point M to chord AB such that $AM = BM$ and draw the line-segment OA and OB.

In $\triangle AMO$ and $\triangle BMO$

$$OA = OB \qquad (Radii\ of\ circle)$$

$$OM = OM \qquad (Common\ side)$$

$$AM = BM \qquad (M\ mid\ point\ of\ AB)$$

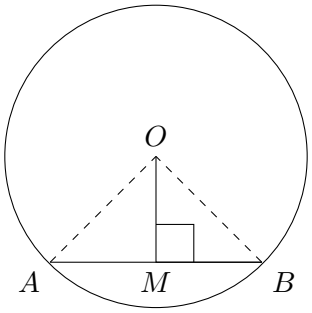



Figure 9.4.2

So,

$$\triangle AMO \cong \triangle BMO \qquad (RHS\ Rule)$$

Hence,

$$\angle AMO = \angle BMO \qquad (CPCT)$$

$$\angle AMO + \angle BMO = 180^\circ \qquad (Linear\ pair\ angle)$$

Hence, $\angle AMO = \angle BMO = 90^\circ$. Therefore, $OM \perp AB$. □

**Definition 9.4.3.** The length of the perpendicular from a point to a line is the **distance of the line from the point**.

**Theorem 9.4.4:** *Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).*

*Proof.* In Fig. (9.4.3), we draw a circle with centre O and two equal chord PQ and RS.

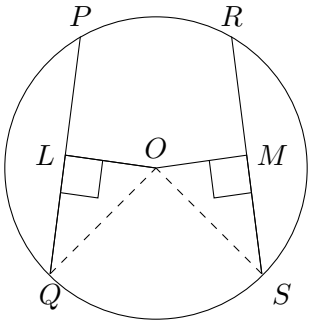



Figure 9.4.3

The points L and M are mid points of chords PQ and RS respectively.

Now in $\triangle QLO$ and $\triangle SMO$

$$OQ = OS \qquad (Radii\ of\ circle)$$

$$LQ = MS \qquad (Mid\ points\ of\ chords\ PQ\ and\ RS)$$

$$\angle QLO = \angle SMO \qquad (Right\ angle)$$

So,

$$\triangle QLO \cong \triangle SMO \qquad (RHS\ rule)$$

Hence,

$$OL = OM \qquad (CPCT)$$

The chords PQ and RS are equal distance from centre O. □

**Theorem 9.4.5:** *Chords equidistant from the centre of a circle are equal in length.*

*Proof.* In Fig. (9.4.4), we draw a circle with centre O and two chord PQ and RS which are equidistant from centre O. So,

$$OL = OM$$

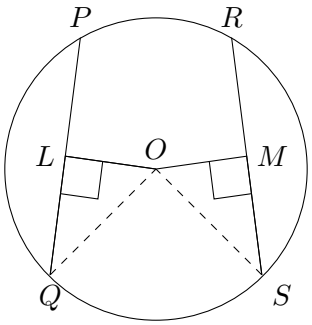



Figure 9.4.4

The points L and M are mid points of chords PQ and RS respectively.

Now in $\triangle QLO$ and $\triangle SMO$

$$OQ = OS \qquad (Radii\ of\ circle)$$

$$LO = MO \qquad (Proved)$$

$$\angle QLO = \angle SMO \qquad (Right\ angle)$$

So,

$$\triangle QLO \cong \triangle SMO \qquad (RHS\ rule)$$

Hence,

$$QL = SM \qquad (CPCT)$$

We get,

$$2QL = 2SM \implies PQ = RS$$

Therefore, chords PQ and RS are equal in length. □

**Example 9.4.6.** If two intersecting chords of a circle make equal angles with the diameter passing through their point of intersection, prove that the chords are equal.

**Solution :** The two chords AB and CD of circle with centre O intersecting at the point E. PQ is a diameter passing through point E, such that $\angle AEQ = \angle DEQ$ as in Fig. (9.4.5). We will prove that $AB = CD$.

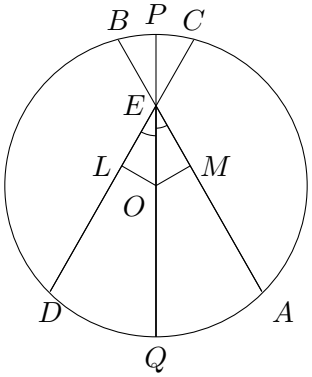



Figure 9.4.5

Draw the perpendicular OL and OM to chord CD and AM from centre O

respectively. Given that $\angle AEQ = \angle DEQ \implies \angle MOE = \angle LEO$. Now,

$$\angle LEO = 180^\circ - 90^\circ - \angle LEO = 90^\circ - \angle LEO \ (Angle\ sum\ property\ of\ triangle)$$

$$= 90^\circ - \angle MEO = \angle MOE$$

Now, in $\triangle LEO$ and $\triangle MEO$,

$$\angle LOE = \angle MOE \qquad (Proved)$$

$$\angle LEO = \angle MEO \qquad (Given)$$

$$OE = OE \qquad (Common)$$

So,

$$\triangle LEO \cong \triangle MEO \qquad (ASA\ rule)$$

Therefore,

$$OL = OM \qquad (CPCT)$$

$$AB = CD \qquad (Theorem\ (9.4.5))$$

## 9.5 Solution of Exercise 9.2

**Problem 9.5.1.** Two circle of radii 5 $cm$ and 3 $cm$ intersect at two points and the distance between their cetres is 4 $cm$. Find the length of the common chord.

**Solution :** In Fig. (9.5.1), two circles of radii 5cm and 3cm with centre O and O' intersect at two points P and Q.

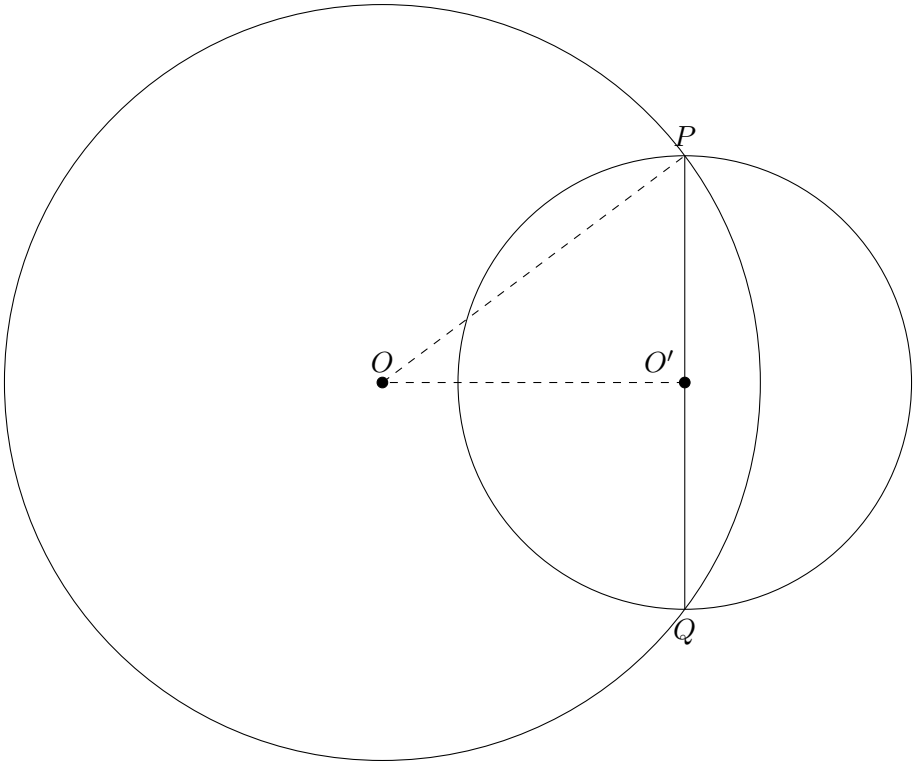



Figure 9.5.1

The common chord PQ will be perpendicular to line-segment OO'. The centre of small circle lies on chord. So, the $\triangle POO'$ is right angle triangle.

$$OP^2 = O'O^2 + O'P^2 \implies O'P^2 = 5^2 - 4^2$$

$$O'P^2 = 25 - 16 = 9$$

$$O'P = 3$$

Hence, length of chord $= 2O'P = 3\ cm$.

**Problem 9.5.2.** If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.

**Solution :** The two chords AB and CD of circle with centre O intersecting at the point E. PQ is a diameter passing through point E, such that $AB = DC$ as in Fig. (9.5.2). We will prove that $AE = DE$ and $BE = CE$.

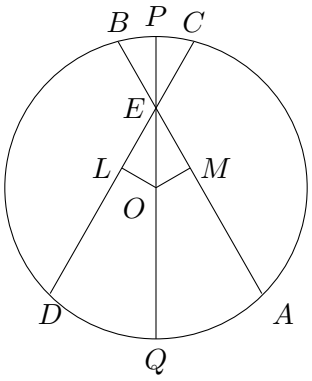



Figure 9.5.2

Draw the perpendicular OL and OM to chord CD and AM from centre O respectively. So, $\angle OME$ and $\angle OLE$ are right angle. Now, in $\triangle LEO$ and $\triangle MEO$,

$$\angle ELO = \angle OME \qquad (Right\ angle)$$

$$LO = MO \qquad (Perpendicular\ to\ equal\ chord)$$

$$OE = OE \qquad (Common)$$

So,

$$\triangle LEO \cong \triangle MEO \qquad (RHS\ rule)$$

Therefore,

$$OL = OM \qquad (CPCT)$$

$$AM = DL \qquad (Theorem\ (9.4.2))$$

$$EL = EM \qquad (CPCT)$$

Therefore,

$$AM + ME = DL + LE \implies AE = DE$$

Similarly,

$$BE = CE$$

**Problem 9.5.3.** If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angle with chords.

**Solution :** The two chords AB and CD of circle with centre O intersecting at the point E. PQ is a diameter passing through point E, such that $AB = DC$ as in Fig. (9.5.2). We will prove that $\angle OEL = \angle OEM$.

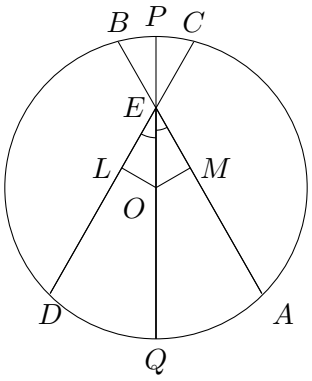



Figure 9.5.3

Draw the perpendicular OL and OM the chord CD and AM from centre O respectively. So, $\angle OME$ and $\angle OLE$ are right angle. Now, in $\triangle LEO$ and $\triangle MEO$,

$$\angle ELO = \angle OME \qquad (Right\ angle)$$

$$LO = MO \qquad (Proved)$$

$$OE = OE \qquad (Common)$$

So,

$$\triangle LEO \cong \triangle MEO \qquad (RHS\ rule)$$

Therefore,

$$\angle OEL = \angle OEM \qquad (CPCT)$$

**Problem 9.5.4.** If a line intersects two concentric circles (circle with same centre) with centre O at A, B, C and D, prove that $AB = CD$ as in Fig. (9.5.4).

**Solutiom :** In Fig. (9.5.4), a common chord of two concentric circle with centre O passing through points A, B, C and D. Draw OL perpendicular to BC.

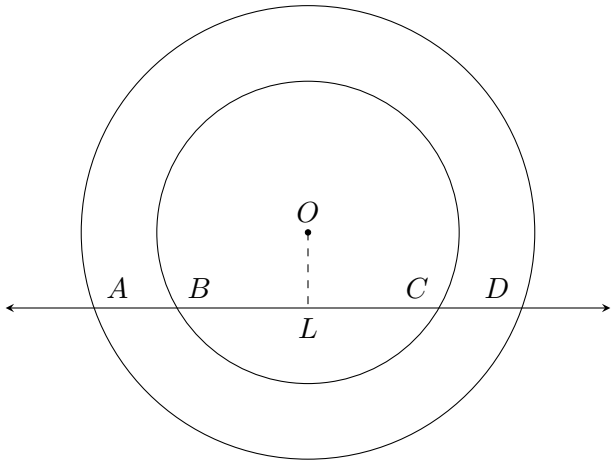



Figure 9.5.4

The line $OL \perp BC$ and $OL \perp AD$. Therefore,

$$BL = CL \qquad (Theorem\ 9.4.1)$$

$$AL = DL \qquad (Theorem\ 9.4.1)$$

Hence,

$$AL - BL = DL - CL \qquad (Above\ condition)$$

$$AB = CD.$$

**Problem 9.5.5.** Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5 $m$ drawn in a park. Reshma throws a ball to Salma, Salma to mandip, Mandip to Reshma. If the the distance between Reshma and Salma and between Salma and mandip is 6 $m$ each, what is the distance between Reshma and Mandip?

**Solution :** In Fig. (9.5.5), Reshma, Salma and Mandi standing on circle at points R, S and M respectively. The points L, N amd P are mid points between Reshma and Mandip and between Reshama and Salma and between Salma and Mandip respectively.

Draw the perpendicular OL, ON and OP from O to line-segment RM, RS and SM. Join the points R, S and M with O.

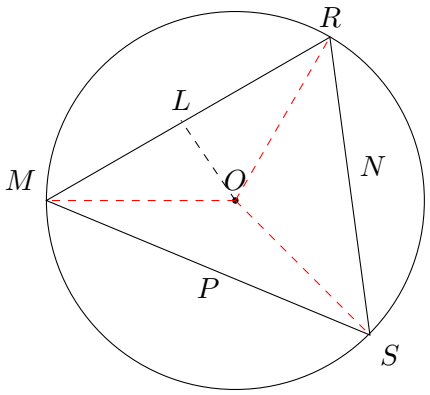



Figure 9.5.5

Distance from Reshma to Salama and Salma to mandip is 12 $cm$. So,

$$RS = 6\ cm\ SM = 6\ cm \qquad (Given)$$

Now in $\triangle MSR$,

$$RS = SM$$

So, it is isoscles triangle and perpendicular from point S to line segment MR, LS will be perpendicular bisector of MR. Let $ML = x,\ OL = y$ So,

$$y^2 + x^2 = 5^2 \implies y^2 + x^2 = 25$$

$$(y+5)^2 + x^2 = 6^2 \implies y^2 + 10y + 25 + x^2 = 36$$

Because $\triangle OLM$ and $\triangle SLM$ are right angle triangle. Now from above two equation,

$$10y + 25 = 11 \implies 10y = -14 \implies y = -1.4$$

$$(-1.4)^2 + x^2 = 25 \implies x = 4.83cm$$

So $MR = 2x = 9.6cm$

**Problem 9.5.6.** A circular park of radius 20 $m$ is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

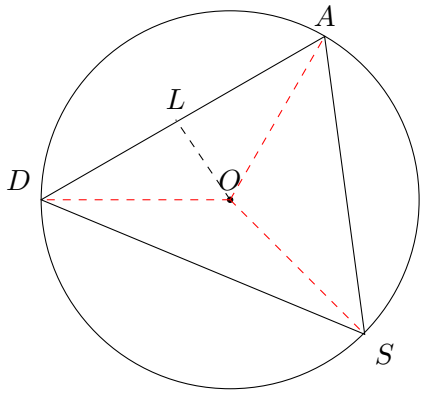



Figure 9.5.6

**Solution:** Now the Ankur, Syed and David sitting at points A, S and D on boundary of circular path of radius 20 $cm$. Now the distance AS, SD and DA are equal. Let $AS = x$ $OL = y$, SL is perpendicular on DA, So,

$$\frac{x^2}{4} + y^2 = 20^2 = 400$$

and

$$(20 + y)^2 + \frac{x^2}{4} = x^2$$

From above two equations

$$400 + 40y + y^2 + 400 - y^2 = 1600 - 4y^2$$

$$4y^2 + 40y - 800 = 0 \implies y^2 + 10y - 200 = 0 \implies (y + 20)(y - 10) = 0$$

Hence $y = 10$, Now $\frac{x^2}{4} = 400 - 100 = 300 \implies x^2 = 1200 \implies x = 34.64$.

Hence required length is 103.92 $m$.

## 9.6 Angle Subtended by an Arc of a Circle

Any part of circumference of circle is an arc.

" *If two chords of a circle are equal, then their corresponding arcs are congruent and conversely, if two arcs are congruent, then their corresponding chords are equal,*"

**Result:** *Congruent arcs (or equal arcs) of a circle subtend equal angles at the centre.*

**Theorem 9.6.1:** *The angle subtended by an arc at the centre is double the angle subtend by it at any point on the remaining part of the circle.*

*Proof.* In Fig. (9.6.1), given an arc PQ of a circle subtend $\angle POQ$ at the centre O and $\angle PAQ$ at a point A on the remaining part of the circle. To prove theorem, we need to prove $\angle POQ = 2\angle PAQ$.

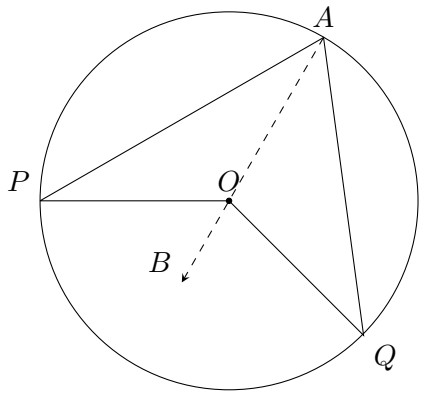



Figure 9.6.1

Let us begin by joining OA and extending it to a point B.

$$\angle BOQ = \angle OAQ + \angle AQO$$

because an exterior angle of a triangle is equal to the sum of the two interior opposit angle. In $\triangle AOQ$,

$$OA = OQ \qquad (Radii\ of\ acircle)$$

So,

$$\angle OAQ = \angle OQA \qquad (\triangle AOQ\ is\ an\ isoscele\ triangle)$$

Hence,

$$\angle BOQ = 2\angle OAQ \tag{9.6.1}$$

Similarly,

$$\angle BOP = 2\angle OAP \tag{9.6.2}$$

From equation (9.6.1) and (9.6.2),

$$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$$

This gives,

$$\angle POQ = 2\angle PAQ \tag{9.6.3}$$

From equation (9.6.3), reflex $\angle POQ = 2\angle PAQ$. □

**Theorem 9.6.2:** *Angle in the same segment of a circle are equal.*

*Proof.* In Fig. (9.6.2), given an arc PQ of a circle subtend $\angle POQ$ at centre O and $\angle PAQ$ and $\angle PCQ$ at points A and C on the remaining part of the circle. To prove theorem, we need to prove $\angle PAQ = \angle PCQ$.

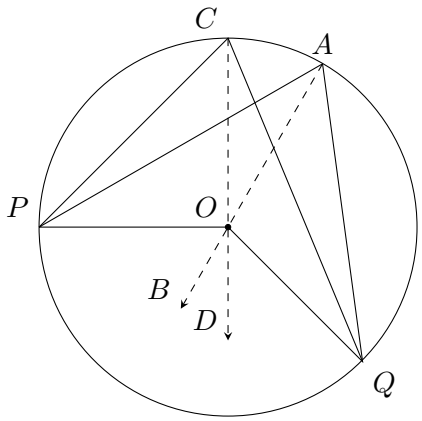



Figure 9.6.2

We have,

(9.6.4) $\angle POQ = 2\angle PAQ$ *By theorem* (9.6.1)

Similarly,

(9.6.5) $\angle POQ = 2\angle PCQ$ *By theorem* (9.6.1)

From equation (9.6.4) and (9.6.5),

(9.6.6) $\angle PAQ = \angle PCQ$

So, angle in the same segment of a circle are equal. □

**Note :** Similarly we can prove that the angle in a semicircle is a right angle.

**Theorem 9.6.3:** *If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic).*

*Proof.* Consider in Fig. (9.6.3), AB is a line segment, which subtends equal angle angles at two points C and D. Therefore,

$$\angle ACB = \angle ADB \qquad (Given)$$

We need to prove that A, B, C and D lie on a circle.

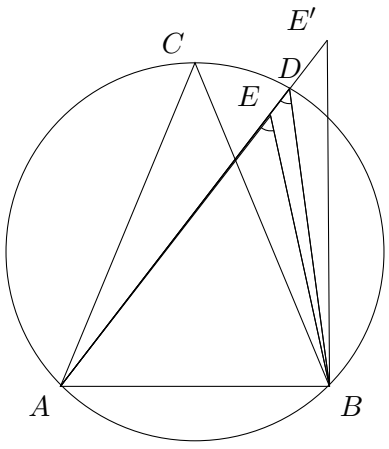



Figure 9.6.3

Let us draw a circle through the points A, C and B. Suppose it does not pass through the point D. Then it will intersect AD (or extended AD) at a point, say E (or E').

Now consider points A, C, E and B lie on a circle, then

$$\angle ACB = \angle AEB \qquad (Lie\ same\ segment)$$

But we have given that,

$$\angle ACB = \angle ADB$$

Therefore,

$$\angle AEB = \angle ADB.$$

It will be possible only when E coincides with D. Similarly, E' should also coincide with D. Hence A, B, C and D lie on circle. □

## 9.7 Cyclic Quadrilaterals

**Definition 9.7.1.** A quadrilateral ABCD ia called *cyclic* if all four vertices of its lie on a circle.

**Theorem 9.7.2:** *The sum of either pair of opposite angles of a cyclic quadrilateral is* 180°.

*Proof.* Consider in Fig. (9.7.1), the quadrilateral ABCD is a cyclic quadrilateral.

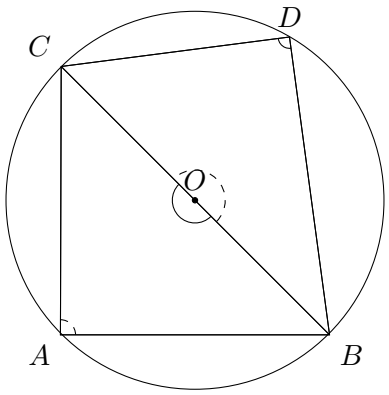



Figure 9.7.1

(9.7.1) $\angle COB = 2\angle CDB$ *Theorem* (9.6.1)

Similarly,

(9.7.2) $\angle BOC = 2\angle BAC$ *Theorem* (9.6.1)

From equation (9.7.1) and (9.72),

$$\angle COB + \angle BOC = 2(\angle CDB + \angle BAC)$$

$$2(\angle CDB + \angle BAC) = 360^\circ \implies \angle CDB + \angle BAC = 180^\circ$$

Similarly,

$$\angle ACD + \angle DBA = 180^\circ$$

□

**Theorem 9.7.3:** *If the sum of a pair of opposite angles of a quadrilateralis* 180°*, the quadrilateral is cyclic.*

*Proof.* In Fig. (9.7.2), ABCD is an quadrilateral in which the sum of a pair of opposite angles is 180°.

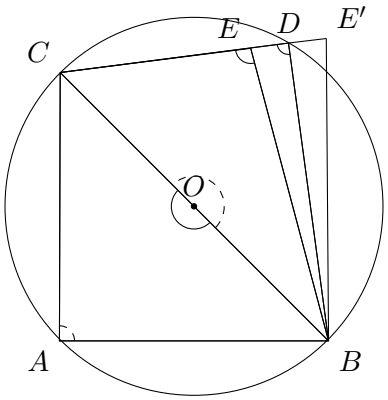



Figure 9.7.2

We need to prove A, B, C and D lie on circle.

Let us draw a circle passing through points A, B and C. Supose it does not pass through poin D. Then it intersect CD (or extended CD ) at a point, say E (or E').

Now consider A, B, C and E lie on a circle, then

$$\angle BAC + \angle CEB = 180^\circ$$

But we have,

$$\angle BAC + \angle CDB = 180^\circ$$

Therefore,

$$\angle CEB = \angle CDB$$

This is not possible unless E coincides D. Because line-segment from diffrent points E and D on same line with point B can not makes same angle at E and D.

Similary, E' coincide D. Hence A, B, C and D lie on circle. □

**Example 9.7.4.** In Fig. (9.7.3), AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60^\circ$

**Solution :** Join OC, OD and BC. The $\triangle COD$ is equilateral. Because chord CD is equal to radius OC and OD. Therefore,

$$\angle COD = 60^\circ$$

Now,

$$\angle CBD = \frac{1}{2}\angle COD \qquad (Theorem\ 9.6.1)$$

We get

$$\angle CBD = 30^\circ$$

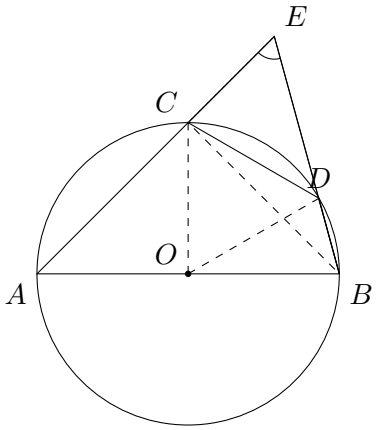



Figure 9.7.3

Next,

$$\angle ACB = 90^\circ \qquad (Angle\ of\ semicircle)$$

So,

$$\angle BCE = 180^\circ - \angle ACB = 90^\circ$$

Hence,

$$\angle CEB = 90^\circ - 30^\circ = 60^\circ$$

**Example 9.7.5.** In Fig. (9.7.4), ABCD is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55^\circ$ and $\angle BAC = 45^\circ$, find $\angle BCD$.

**Solution :** In Fig. (9.7.4),

$$\angle DBC = \angle DAC = 55^\circ \qquad (Angle\ in\ the\ same\ segment)$$

Therefore,

$$\angle DAB = \angle CAD + \angle BAC = 55^\circ + 45^\circ = 100^\circ$$

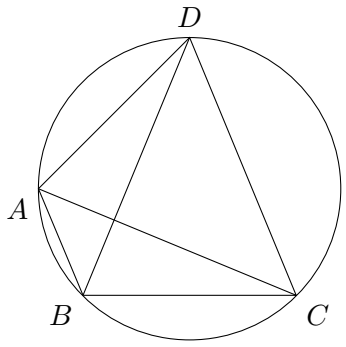



Figure 9.7.4

But,

$\angle DAB + \angle BCD = 180^\circ$ (*Opposite angle of a cyclic quadrilateral*)

So,

$$\angle BCD = 180^\circ - 100^\circ = 80^\circ$$

**Example 9.7.6.** Two circles intersect at two points A and B. AD and AC are diameters to the two circles as in fig. (9.7.5). Prove that B lies on the line segment DC.

**Solution :** Join AB.

$\angle ABD = 90^\circ$ (*Angle in a semicircle*)

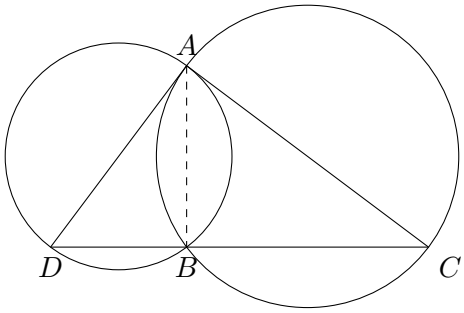



Figure 9.7.5

$$\angle ABC = 90^\circ \qquad (\textit{Angle in a semicircle})$$

So,

$$\angle ABD + \angle ABC = 90^\circ + 90^\circ = 180^\circ$$

Therefore, DBC is a line. It means B lie on the line segment DC.

**Example 9.7.7.** Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

**Solution :** In Fig. (9.7.6), ABCD is a quadrilateral in which the angle bisectors AH, BF, CF and DH of internal angle A, B, C and D respectively form a quadrilateral EFGH.

Now,

$$\angle FEH = \angle AEB = 180^\circ - \angle EAB - \angle EBA \qquad (\textit{Vertically opposite angle})$$

$$= 180^\circ - \frac{1}{2}(\angle A + \angle B)$$

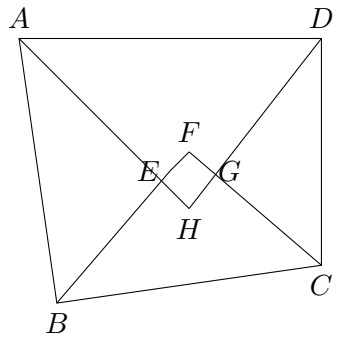



Figure 9.7.6

and Now,

$$\angle FGH = \angle CGD = 180^\circ - \angle GCD - \angle GDC \qquad (Vertically\ opposite\ angle)$$

$$= 180^\circ - \frac{1}{2}(\angle C + \angle D)$$

Therefore,

$$\angle FEH + \angle FGH = 180^\circ - \frac{1}{2}(\angle A + \angle B) + 180^\circ - \frac{1}{2}(\angle C + \angle D)$$

$$= 360^\circ - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 360^\circ - \frac{1}{2} \times 360^\circ$$

$$= 360^\circ - 180^\circ = 180^\circ$$

Therefore, by Theorem (9.7.3), the quadrilateral EFGH is cyclic.

## 9.8 Solution of Exercise 9.3

**Problem 9.8.1.** In Fig.(9.8.1), A, B and C are three points on a circle with centre O such that $\angle BOC = 30^\circ$ and $\angle AOB = 60^\circ$. If D is a point on the

circke other than the arc ABC, find $\angle ADC$.

**Solution:** In the Fig. (9.8.1),the points A, B, C and D on a circle with centre O such that $\angle COB = 30^\circ$ and $\angle BOA = 60^\circ$.

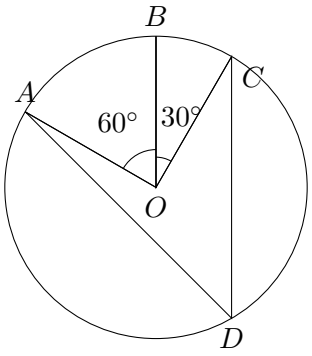



Figure 9.8.1

According the Theorem (9.6.1),

$$\frac{1}{2}\left(\angle COB + \angle BOA\right) = \angle ADC$$

$$\angle ADC = \frac{1}{2}\left(60^\circ + 30^\circ\right) = \frac{90^\circ}{2} = 45^\circ$$

.

**Problem 9.8.2.** A chord of a circle is equal to the radius of the circle. Find the angle sustended by the chord at a point on the minor arc and also at a point on the major arc.

**Solution:** In Fig. (9.8.2), AB a chord of circle is equal radius of circle at centre O with C and D points on corresponding minor arc and major arc of circle.

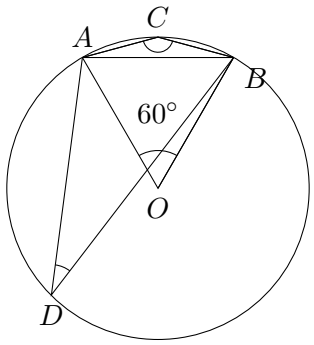



Figure 9.8.2

Now, $AB = AO = OB = Radius$. Therefore $\triangle AOB$ is an equilateral triangle so every angle of triangle equal to $60°$.

According theorem (9.6.1),

$$\angle ADB = \frac{1}{2}\angle AOB = \frac{60°}{2} = 30°$$

Similarly

$$\angle ACB = \frac{1}{2}(Reflex\,angle \angle AOB)$$

Next,

$$\angle ADB + \angle ACB = \frac{1}{2}\angle AOB + \frac{1}{2}(Reflex\,angle \angle AOB)$$

Therefore $\angle ADB + \angle ACB = \frac{1}{2} \times 360° = 180°$, So, $\angle ACB = 180° - 30° = 150°$

Another way in Fig. (9.8.2), $\square ACBD$ is a cyclic quadrilateral. The sum of opposite angle of cyclic quadrilateral is $180^\circ$.

**Problem 9.8.3.** In Fig. (9.8.3), $\angle ACB = 100^\circ$, where A, C and B are points on circle with centre O. Find $\angle OAB$.

**Solution:** In Fig. (9.8.3), arc AB substend $\angle ACB$ at minor arc and $Reflex\, angle \angle AOB$ at the centre. So according theorem (9.6.1), $Reflex\ angle\ \angle AOB = 2\angle ACB = 2 \times 100^\circ = 200^\circ$.

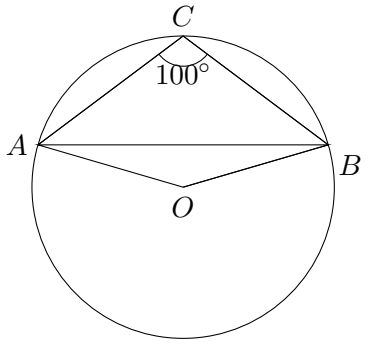



Figure 9.8.3

Therefore, $\angle AOB = 360^\circ - reflex\ angle\ \angle AOB = 360^\circ - 200^\circ = 160^\circ$.

Now in $\triangle AOB$, $\angle OAB = \angle OBA$, because side $OA = OB = Radius$. Next

$$\angle OAB + \angle OBA + \angle AOB = 180^\circ$$

$$\angle OAB + \angle OBA = 180^\circ - \angle AOB = 180^\circ - 160^\circ = 20^\circ$$

$$2\angle OAB = 20^\circ \implies \angle OAB = 10^\circ.$$

**Problem 9.8.4.** In Fig. (9.8.4), $\angle DAB = 69°$, $\angle DBA = 31°$, find $\angle ACB$.

**Solution:** In the Fig. (9.8.4), $\angle DAB = 69°$ and $\angle DBA = 31°$

$$\angle ADB = \angle ACB \ \textit{Lies in same circle segment}$$

Now in $\triangle ADB$

$$\angle DAB + \angle DBA + \angle ADB = 180°$$

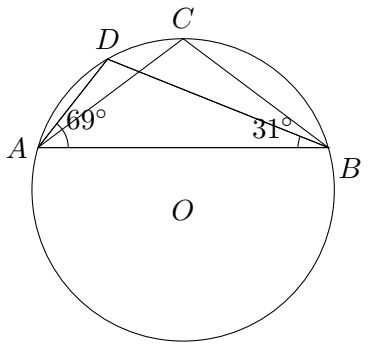



Figure 9.8.4

$\angle ADB = 180° - (69° + 31°) = 80°$ So, $\angle ACB = \angle ADB = 80°$.

**Problem 9.8.5.** In Fig. (9.8.5), A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130°$ and $\angle ECD = 20°$. Find $\angle BAC$.

**Solution:** In Fig. (9.8.5), $\angle BEC = 130^\circ$ and $\angle ECD = 20^\circ$. The points A and D on monor arc of circle, so

$$\angle BAC = \angle BDC, \ \ Theorem \ (9.6.2)$$

$$\angle BEC + \angle CED = 180^\circ, \ \ Linear \ \ pair$$

So,

$$\angle CED = 180^\circ - 130^\circ = 50^\circ$$

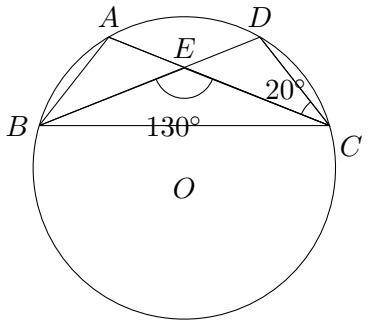



Figure 9.8.5

Now in $\triangle CED$

$$\angle EDC = 180^\circ - 20^\circ - 50^\circ = 110^\circ$$

So

$$\angle BAC = \angle BDC = \angle EDC = 110^\circ.$$

**Problem 9.8.6.** ABCD ia a cyclic quadrilateral whose digonal intersect at a point E. If $\angle DBC = 70°$, $\angle BAC = 30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.

**Solution:** In Fig. (9.8.6), $\angle DBC = 70°$, $\angle BAC = 30°$. Here,

$$\angle CBD = \angle CAD \ \ in \ (segment \ of \ circle)$$

So, $\angle CAD = 70°$

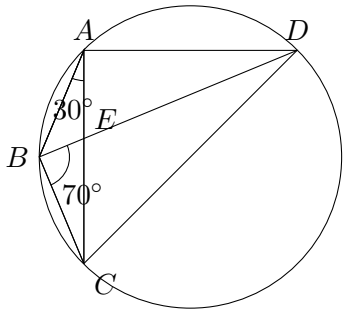



Figure 9.8.6

$$\angle BCD + \angle BAD = 180°(opposit \ \ angle \ of \ cyclic \ quadrilateral)$$

Hence, $\angle BCD = 180° - \angle BAD = 180° - 30° - 70° = 80°$. Because

$$\angle BAC = \angle BDC \quad (in \ same \ circle \ segmaent)$$

IF $AB = BC$ then $arc \ AB = arc \ BC$, so

$$\angle ADB = \angle BDC \tag{9.8.1}$$

and

$$\angle BCA = \angle BDA \tag{9.8.2}$$

and

$$\angle BDC = \angle BAC = 30^\circ \tag{9.8.3}$$

Form equation (9.8.1), (9.8.2) and (9.8.3) we get,

$$\angle BCA = 30^\circ$$

Now $\angle ECD = \angle BCD - \angle BCA = 80^\circ - 30^\circ = 50^\circ$.

**Problem 9.8.7.** If diagonal of a cyclic quadrilateral are diameter of the circle through the vertics of the quadrilateral, prove that it is a rectangle.

**Solution:** In Fig.(9.8.7), $\square ABCD$ is cyclic quadrilateral and diagonal BD and AC are diameter of circle.

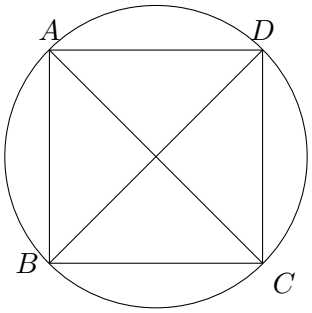



Figure 9.8.7

Now the $AC$ is diameter of circle then $\angle ADC = \angle ABC = 90°$, because they are angle of semicircle.

Similarly $BD$ is diameter of circle so, $\angle BAD = \angle BCD = 90°$. In $\square ABCD$ each angle of quadrilateral are right angle so, $ABCD$ is rectangle.

**Problem 9.8.8.** If the non-parallel sides of trapezium are equal, prove that it is cyclic.

**Solution:** In Fig. (9.8.8), $\square ABCD$ is a trapezium with $AB = DC$ and $AD \parallel BC$. Draw perpendicular from point B and C to AD. We get two right angle triangle $\triangle ABP$ and $\triangle CQD$.

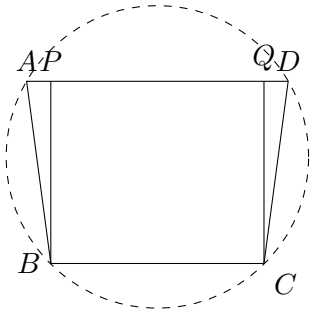



Figure 9.8.8

Now, in $\triangle ABP$ and $\triangle CQD$.

$$AB = DC \quad (Given)$$

$$BP = CQ \quad (distance\ between\ parallel\ line)$$

$$\angle BPA = \angle CQD = 90°$$

Hence,

$$\triangle BAP \cong \triangle CDQ \quad (RHS\ rule)$$

So,

$$\angle BAP = \angle CDQ \quad (CPCT)$$

$$\angle QDC + \angle DCB = 180^\circ \quad (Co - interier\ angle)$$

So, $\angle BAP + \angle DCB = 180^\circ \implies \angle BAD + \angle DCB = 180^\circ$.

Similarly,$\angle CDQ + \angle ABC = 180^\circ \implies \angle CDA + \angle ABC = 180^\circ$ Hence $\square ABCD$ is a cyclic quadrilateral.

**Problem 9.8.9.** Two circle intersect at two points B and C. Through B two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively (see Fig 9.8.9). Prove that $\angle ACP = \angle QCD$.

**Solution:** In Fig. (9.8.9), ABD and PBQ two lines segment passing through point B. The end points P,A and D, Q on the circle.

$$\angle ABP = \angle DBQ \qquad vertically\ opposite\ angle$$

Next

$$\angle PBA = \angle PCA \qquad lies\ on\ same\ segment\ of\ circle$$

Similarly,

$$\angle DBQ = \angle DCQ \qquad lies\ on\ same\ segment\ of\ circle$$

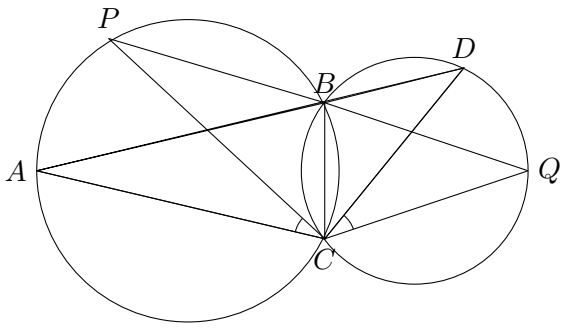



Figure 9.8.9

Hence, from above three conditions, get

$$\angle PCA = \angle DCQ.$$

**Problem 9.8.10.** If circle are taking two sides of a triangle as diameters, prove that the point of intersection of these circle lie on the third side.

**Solution:** In Fig. (9.8.10), AC and CD are diameters of two circle. The intersection of these two circle is point B, draw a line segment BC.

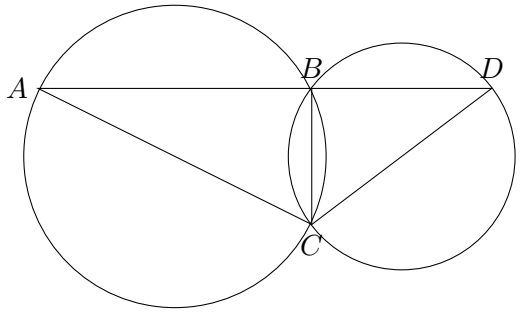



Figure 9.8.10

Now in $\triangle ABC$,

$$\angle ABC = 90^\circ \qquad angle\ of\ semicircle$$

Similarly, in $\triangle BCD$,

$$\angle CBD = 90^\circ \qquad angle\ of\ semicircle.$$

Hence,

$$\angle ABD = \angle ABC + \angle CBD = 90^\circ + 90^\circ = 180^\circ.$$

Therefore, ABD is a line segment which is passes through the point of intersection of two circles.

**Problem 9.8.11.** ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.

**Solution:** In Fig. (9.8.11), ABC and ADC are right angle triangle with common hypotenuse AC.

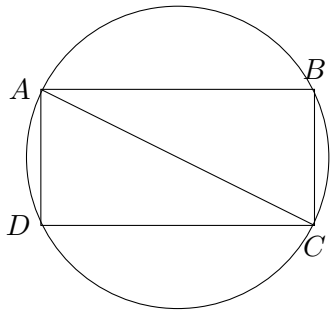



Figure 9.8.11

We can draw a circle as diameter AC passes through the vertex of triangle ABC and ADC. The $\angle CAD$ and $\angle CBD$ are same side of chord DC. Hence

$$\angle CAD = \angle CBD \qquad angle\ of\ same\ circle\ segment.$$

**Problem 9.8.12.** Prove that a cyclic parallelogram is a rectangle.

**Solution:** In Fig. (9.8.12), $\square ABCD$ is a parallelogram, therefore opposite angle will be equal.

$$\angle ADC = \angle ABC \qquad Opposite\ angle\ of\ parallelogram$$

Now, the $\square ABCD$ is a cyclic quadrilateral therefore, sum of opposite angle

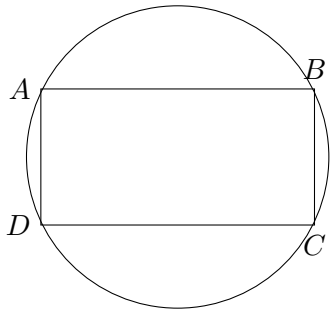



Figure 9.8.12

will be $180^\circ$.

$$\angle ABC + \angle ADC = 180^\circ$$

So,

$$2\angle ABC = 180^\circ \implies \angle ABC = 90^\circ.$$

Similaraly,

$$\angle ABC = \angle BCD = \angle CDA = \angle DAC = 90^\circ.$$

Hence, $\square ABCD$ is a rectangle.

# 10 HERON'S FORMULA

## 10.1 Introduction:

We know that area of a triangle is $= \frac{1}{2} \times base \times height$, but if the hight of triangle is unknown then we can not use the known farmula. The formula that gives area of any triangle which sides are known is written as;

$$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $a, b$ and $c$ are sides of the triangle, and

$$s\text{= semi-perimeter } s = \frac{a+b+c}{2}$$

This formula known as Heron's formula. Because this farmula has derived by Heron who was born in about 10 $AD$ possibaly in Alexandria in Egipt.

**Example 10.1.1.** Find the area of a triangle, two sides of which are 8 $cm$ and 11 $cm$ and perimeter is $32cm$.

**Solution :** We have perimeter of the triangle $= 32$ $cm$ $a = 8$ $cm$ and $b =$ 11 $cm$. In Fig. (10.1.1), side $a = 8$ $cm$, $b = 11$ $cm$ and perimeter $2s = 32$ $cm$.

Now, $c = 2s - a - b = 32 - 8 - 11 = 13cm$ So, $2s = 32 \implies s = 16cm$

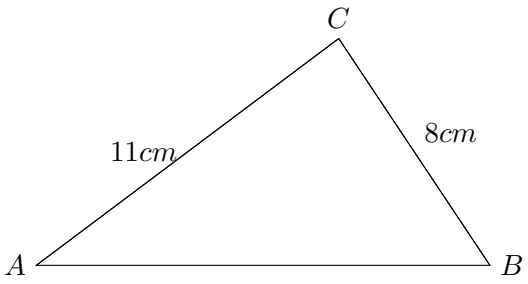



Figure 10.1.1

$$s - a = 16 - 8 = 8cm$$

$$s - b = 16 - 11 = 5cm$$

$$s - c = 16 - 13 = 3cm$$

Hence, area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{16 \times 8 \times 5 \times 3} = 8\sqrt{30}cm^2$$

**Example 10.1.2.** A triangular park ABC has sides 120 $cm$, 80 $cm$ and 50 $cm$. A gardener Dhania has to put a fence all around it and also plant grass inside. How much area does she need to plant ? Find the cost of fencing it with barbed wire at the rate of $rs$ 20 per meter leaving a space $3m$ wide for gate on one side. **Solution :** In Fig. (10.1.2), $a = 120m,\ b = 80m,\ c = 50m$.

We have $2s = 50m + 80m + 120m = 250m$

$$\implies s = 125m$$

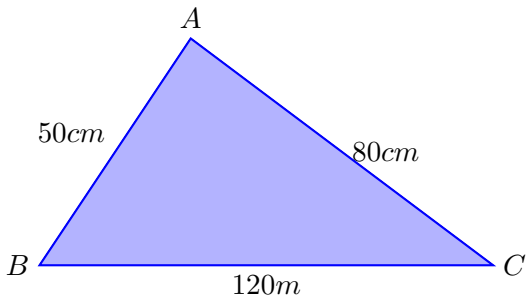



Figure 10.1.2

Now

$$s - a = (125 - 120)m = 5m$$

$$s - b = (125 - 80)m = 45m$$

$$s - c = (125 - 50)m = 75m$$

Therefore, area of the park $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{125 \times 5 \times 45 \times 75}\, m^2$$

$$= 275\sqrt{15}\, m^2.$$

Also perimeter of the park $= AB + BC + CA = 120\, m$. Therefore, length of the wire needed for fencing $= 120m - 3m = 247m$.

So the cost of fencing $= 20 = 4940rs$.

**Example 10.1.3.** The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 $m$. Find its area.

**Solution:** Suppose that the sides, in meters, are $3x, 5x$ and $7x$ because ratio of the sides are $3 : 5 : 7$.

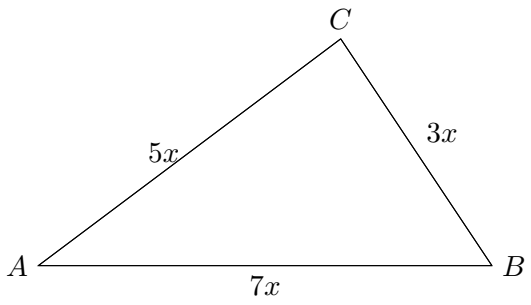



Figure 10.1.3

Then, perimeter of $\triangle ABC = AB + BC + CA = 7x + 3x + 5x = 15x$. We have

$$15x = 300 \implies x = 20.$$

Therefore side of triangle are $3 \times 20\ m$, $5 \times 20\ m$, and $7 \times 20\ m$. or $60m$, $100m$ and $140m$. So,

$$s = \frac{60 + 100 + 140}{2} = 150\ m.$$

Area $= \sqrt{150(150 - 60)(150 - 100)(150 - 140)}\ m^2$

$$= \sqrt{150 \times 90 \times 50 \times 10}$$

$$= 1500\sqrt{3}\ m^2.$$

## 10.2 Solution of Exercise 10.1

**Problem 10.2.1.** A traffic signal board, indicating 'SCHOOL AHEAD' is an equilateral triangle with side $a$. Find the area of the signal board, using Heron's formula. If its perimeter is 180 $m$,what will be the area of the signal board ?

**Solution:**

Let $\triangle ABC$ is signal board of school with side $a$, which is equilateral triangle (See Fig. 10.2.1).

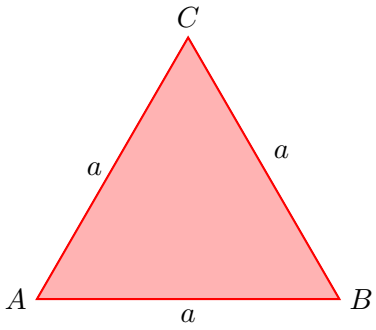



Figure 10.2.1

Here, sides of triangular board are $a$ each. So perimeter of triangular boars is $2s$. According to question

$$2s = a + a + a = 180$$

$$\implies 3a = 180 \implies a = 60$$

Using Heron's formula area of triangular board

$$= \sqrt{90(90-60)(90-60)(90-60)}\ cm^2$$

$$= \sqrt{90 \times 30 \times 30 \times 30}\ cm^2$$

$$= 900\sqrt{3}\ cm^2.$$

**Problem 10.2.2.** The triangular side walls of a flyover have been used for advertisements. The sides of the wall are 122 $m$, 2, $m$ and 120 $m$ (see Fig. 10.2.2). The advertisements yield an earning of rs 5000 per $m^2$ per year. A company hired one of its walls for 3 months. How much rent did it pay ?

**Solution:** In the Fig. (10.2.2), ABC is triangular side wall of a flyover with sides 122 $m$ 22 $m$ and 120 $m$.

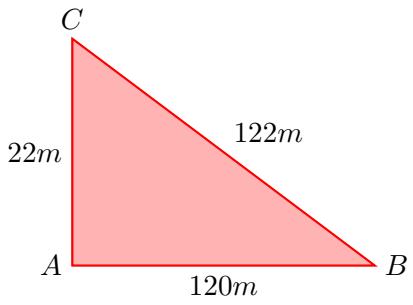



Figure 10.2.2

In triangual wall $a = 122\ m$, $b = 22\ m$ and $c = 120\ m$. So, $2s = 122\ m + 22\ m + 120\ m = 264\ m \implies s = 132\ m$.

Using Heron's formula area of wall

$$= \sqrt{132(132-122)(132-22)(132-120)}\, m^2$$

$$= \sqrt{132 \times 10 \times 110 \times 12}\, m^2$$

$$= 11 \times 10 \times 12\ m^2$$

$$= 1320\ m^2.$$

Rate of rent for advertiesment yearly is rs 5000. Rate of rent for advertiesment monthly $= \frac{5000}{12}\ rs.$

So, rent of advertisement for 3 month

$$= 1320 \times \frac{5000}{12} \times 3$$

$$= 1650000\, rs.$$

**Problem 10.2.3.** There is a slide in park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN" (see Fig. 10.2.3). If the sides of the wall are 15 $m$, 11 $m$ and 6 $m$, find the area painted in colour.

**Solution:** In Fig. (10.2.3), ABCD is a slide whose sides are 15 $m$, 11 $m$ and 6 $m$.

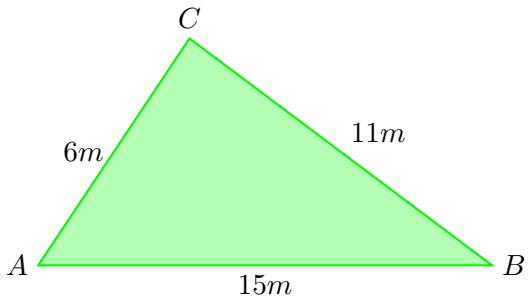



Figure 10.2.3

Dimention of triangular slide are $b = 6\ m$, $a = 11\ m$ and $c = 15\ m$.

So, semiperimeter $s = \frac{6+11+15}{2}\ m = 16\ m$. Using Heron's formula

$$area\ of\ \triangle ABC = \sqrt{16(16-6)(16-11)(16-15)}$$

$$= \sqrt{16 \times 10 \times 5 \times 1} = 20\sqrt{2}\, m^2$$

Hence area painted in colour is $20\sqrt{2}\, m^2$.

**Problem 10.2.4.** Find the area of a triangle two sides of which are $18\, cm$ and $10\, cm$ and the perimeter is $42\, cm$.

**Solution :** Suppose sides of triangle are $a = 18$cm and 10cm and perimeter $2s = 42$cm. Then $s = 41$cm.

So,

$$c = 2s - a - a = 42 - 18 - 10 = 14\, cm$$

Area of triangle

$$= \sqrt{21(21-18)(21-10)(21-14)}$$

$$= \sqrt{21 \times 3 \times 11 \times 7}$$

$$= 21\sqrt{11}\, cm^2.$$

**Problem 10.2.5.** Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540 m. Find its area.

**Solution:** Suppose the side of triangle are $12x$, $17x$ and $25x$. The perimeter is 540 $m$.

According question

$$12x + 17x + 25x = 540\ m$$

$$54x = 540 \implies x = 10.$$

So, sides of triangle are $120m$, 170 $m$ and 250 $m$. Using Heron's formula,

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{270(270-120)(270-170)(270-250)}$$

$$= \sqrt{270 \times 150 \times 100 \times 20}$$

$$= 9000\, m^2$$

**Problem 10.2.6.** An isosceles triangle has perimeter $30\, cm$ and each of the equal sides is $12\, cm$. Find the area of the triangle.

**Solution :** Let two equal sides of triangle are $12\, cm$ each and one side is $a$ cm. The perimeter is 30 cm. So, $s = 15\, cm$.

Therefore,

$$12 + 12 + a = 30 \implies a = 30 - 24 = 6\,cm$$

By Heron's formula

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{15(15-12)(15-12)(15-6)}$$

$$= \sqrt{15 \times 3 \times 3 \times 9}$$

$$= 9\sqrt{15}\,cm^2.$$

# 11 SURFACE AREAS AND VOLUMES

## 11.1 Introduction

In this chapter we will discuss the surface area and volume of right circular cone, sphere. There are mainly three section that interpritation of formula of surface and volume of right circualr cone, sphere.

**Right circular cone :** A shape whose base is a circle and only one vertex and perpendicular line on base with center of circular base passes through vertex. This shape is called *right circular cone.*

The distance of vertex from center of circualr base of cone is called *height* of cone and denoted by $h$. The radius of circular base of cone is called *radius* of cone and denoted by $r$. The distance of vertex from circumference of corcular base is called *slant height* of cone and denoted by $l$.

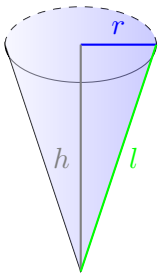



Figure 11.1.1: Cone

If $r$ is radius of right circular cone and $l$ is slant height of cone then, curved

surface area of a cone

$$= \frac{1}{2} \times l \times 2\pi r = \pi r l$$

| Curved surface area of a cone $= \pi r l$ |
|---|

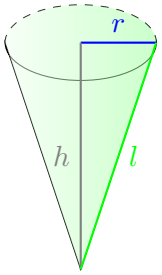



Figure 11.1.2: Cone

From Fig. (11.1.2), $l^2 = r^2 + h^2$, this give

| $l = \sqrt{r^2 + h^2}$. |
|---|

| Area of base of a cone $= \pi r^2$ |
|---|

The total surface area of a cone = curve surface area + area of base.

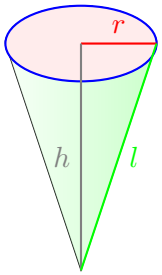



Figure 11.1.3: Cone

| |
|---|
| Total surface area of a cone $= \pi rl + \pi r^2 = \pi r(l + r)$. |

**Example 11.1.1.** Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius is 7 cm.

**Solution :** Curved surface area $= \pi rl$

$$= \frac{22}{7} \times 7 \times 10 \ cm^2$$

$$= 220 \ cm^2.$$

**Example 11.1.2.** The height of a cone is 16 cm and its base radius is 12 cm. Find the curved surface area and total surface area of the cone (Use $\pi = 3.14$).

**Soution :** Here for given cone, $h = 16 \ cm$ and $r = 12 \ cm$.

So, from $l = \sqrt{h^2 + r^2}$, we get

$$l = \sqrt{16^2 + 12^2} = \sqrt{256 + 144} = 20 \ cm.$$

So, curved surface area $= \pi rl$

$$3.14 \times 12 \times 20 \ cm^2$$

$$= 753.6 \ cm^2.$$

Next total suface area $= \pi rl + \pi r^2$

$$= (753.6 + 3.14 \times 12 \times 12) \ cm^2$$

$$= (753.6 + 452.16) \ cm^2$$

$$= 1205.76 \ cm^2.$$

## 11.2 Solution of Exercise 11.1

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

**Problem 11.2.1.** Diameter of the base of a cone is 10.5 cm and its slant height is 10 cm. Find its cueved surface area.

**Solution :** Here, $d = 10.5$ cm and $h = 10$ cm. So, radius $r = 5.25$ cm. Now, the curved surface area of cone $= \pi rl$

$$= \frac{22}{7} \times 5.25 \times 10 \ cm^2$$

$$= 22 \times .75 \times 10 \ cm^2$$

$$= 165 \ cm^2$$

**Problem 11.2.2.** Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m .

**Solution :** The radius of base of cone is $r = \frac{24}{2} = 12\ m$ and slant height is $l = 21$ m. The total surface area of a cone $= \pi r(l + r)$

$$= \frac{22}{7} \times 12\,(21 + 12)\ \ m^2$$

$$= \frac{22}{7} \times 12 \times 32\ m^2$$

$$= 1206.86$$

**Problem 11.2.3.** Curved surface area of a cone is 308 $cm^2$ and its slant height 14 $cm$. Find (i) radius of the base and (ii) total surface area of the cone.

**Solution :** The slant height and curved surface area of the cone are 14 $cm$ and 308 $cm^2$.

We know that, curved surface area of the cone $= \pi rl$.

According question

$$\pi \times r \times 14 = 308 \implies r = \frac{7}{22}\frac{308}{14} = 7\ cm$$

Hence radius of base of the cone is 7 $cm$.

Now, area of base of the cone $= \pi r^2$

$$\pi r^2 = \frac{22}{7} \times 7 \times 7\ cm^2 = 154\ cm^{\cdot}$$

So, total surface area of the cone = curved surface area + area of base.

Total surface area = $308 + 154 = 462\ cm^2$.

**Problem 11.2.4.** A conical tent is 10 $m$ high and the radius of its base is 24 $m$. Find

1. slant height of the tent.
2. cost of the canvas required to make the tent, if the cost of 1 $m^2$ canvas is 70 rs.

**Solution :** The shape of conical tent is right circular cone. So, radius and height of cone are $r = 24\ m$ and $h = 10\ m$. Suppose slant height of cone is $l$.

So, $l = \sqrt{r^2 + h^2}\ m = \sqrt{24^2 + 10^2}\ m$

$$= \sqrt{676}\ m = 26\ m.$$

The slant height of tent is 26m . Now, the canvas required to make tent = To curved surface area of cone. The curved surface area of cone = $\pi rl\ m^2$

$$= \frac{22}{7} \times 24 \times 26\ m^2$$

The cost of canvas require to make tent = $\frac{22}{7} \times 24 \times 26 \times 70 = 137280\ rs$.

**Problem 11.2.5.** What length of tarpaulin 3 m wide will be required to make conical tent of height 8 m and base radius 6 m ? Assume that the extra length

of material that will be required for stiching margins and wastage in cutting is approximately 20 cm (Use $\pi = 3.14$).

**Solution :** The radius and height of cone are $r = 6\ m$ and $h = 8\ m$ .Let slant height is $l$ . So, $l = \sqrt{r^2 + h^2}\ m = \sqrt{6^2 + 8^2}\ m = \sqrt{100}\ = 10\ m$ Further, Curved surface area of the cone $= \pi rl$

$$= 3.14 \times 6 \times 10\ m^2 = 188.4\ m^2.$$

Length of canvas required to make tent = To curved surface area of the cone. Suppose length of canvs is $x$ m. Hence, Area of canvas is equal to curved surface area of tent.

$$x \times 3 = 188.4 \implies x = 62.8\ m$$

Required length of canvas $= (62.8 + .20)\ \ m = 63\ m.$

**Problem 11.2.6.** The slant height and base diameter of a conical tomb are 25 $m$ and 14 $m$ respectively. Find the cost of white-washing its curved surface at the rate of 210 $rs$ per 100 $m^2$.

**Solution :** The diameter of tomb is 14 $m$, so, radius $r = \frac{d}{2} = \frac{14}{2} = 7\ m$. The slant height and base diameter of a conical tomb are $l = 25\ m$ and $r = 7\ m$ respectively. Further, Curved surface area of the cone $= \pi rl$

$$= \frac{22}{7} \times 7 \times 25\ m^2 = 550\ m^2.$$

Rate of white-washing $= \frac{210}{100} = 2.1$ per $m^2$. Cost of white-washing tomb

$$= 550 \times 2.10 = 1155\ rs.$$

**Problem 11.2.7.** A joker's cap is in the form of a right circular cone of base radius 7 $cm$ and height 24 $cm$. Find the area of the sheet required to make 10 such caps .

**Solution :** The radius and height of right circular cone are $r = 7\ cm$ and $h = 24\ cm$. So, $l = \sqrt{r^2 + h^2}\ cm = \sqrt{7^2 + 24^2}\ cm = \sqrt{625}\ cm = 25\ cm$.

Further, Curved surface area of the cone $= \pi rl$

$$= \frac{22}{7} \times 7 \times 25\ cm^2 = 550\ cm^2.$$

The curved surface area of one cap is 550 $cm^2$.

So, area of required sheet for 10 cap

$$= 10 \times 550\ cm^2 = 5500\ cm^2.$$

**Problem 11.2.8.** A bus stop is barricated from the remaining part of the road, by using 50 hollow cones made of recycled cardboard. Each cone has a base diameter of 40 $cm$ and height 1 $m$. If the outer side of each of the cones is to be painted and the cost of painting is 12 $rs$ per $m^2$, what will be the cost of painting all these cones ? (Use $\pi = 3.14$ and take $\sqrt{1.04} = 1.02$ ).

**Solution :** The diameter of conical cardboard is 40 $cm = .40\ m$, so radius of conical cardboard will be $r = \frac{.40}{2}\ m = .20\ m$. The height of conical

cardboard is $h = 1\ m$.

We know that, $l = \sqrt{r^2 + h^2}\ m = \sqrt{(.2)^2 + 1^2}\ m = \sqrt{1.04}\ m = 1.02\ m$.

Next, Curved surface area of the cone $= \pi rl$

$$= 3.14 \times (.2) \times 1.02\ m^2 = 0.64056\ m^2.$$

The curved surface area of one conical cardboard is $0.64056\ cm^2$.

So, area of 50 conical cardboard

$$= 50 \times 0.64056\ m^2 = 32.028\ m^2.$$

Cost of painting 50 cardboard

$$= 12 \times 32.028 = 384.33\ rs.$$

## 11.3 Surface Area of a Sphere

**Sphere :** *A sphere is a three dimentional figure (solid figure), which is made up of all points in the space, which lie at a constant distance called the radius, from a fixed point called the center of the sphere.* The radius and centre are denoted by latter $r$ and $O$ respectively.

**Note** A sphere is like the surface of a ball. The word solid sphere is used for the solid whose surface is a sphere.

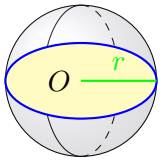



Figure 11.3.1: Sphere

Surface area of a sphere $= 4\pi r^2$

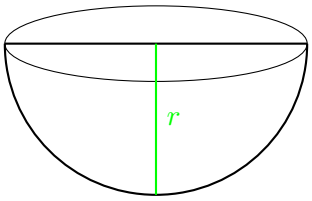



Figure 11.3.2: Sphere

Curved surface area of a hemisphere $= 2\pi r^2$

where $r$ is the radius of the sphere of which the hemisphere is a part.

Now,

Total surface area of a hemisphere $= 3\pi r^2$

**Example 11.3.1.** Find the surface area of a sphere of radius 7 cm.

**Solution :** The radius of sphere is 7 cm. The surface area of a sphere of radius 7 cm would be

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \ cm^2 = 616 \ cm^2.$$

**Example 11.3.2.** Find (i) the curved surface area and (ii) the total surface area of a hemisphere of radius 21 cm.

**Solution :** (i) The curved surface area of a hemisphere of radius 21 cm would be

$$= 2\pi r^2 = 2 \times \frac{22}{7} \times 21 \times 21 \ cm^2 = 2772 \ cm^2.$$

(ii) The total surface area of the hemisphere would be

$$= 3\pi r^2 = 3 \times \frac{22}{7} \times 21 \times 21 \ cm^2 = 4158 \ cm^2.$$

**Example 11.3.3.** The hollow sphere, in which the circus motorcyclist performs his stunts, has a diameter of 7 m. Find the area available to the motorcyclist for riding.

**Solution :** Diameter of the sphere = 7 m. Therefore, radius is 3.5 m. So, the riding space available for the motorcyclist is the surface area of the 'sphere' which is given by

$$4\pi r^2 = 4 \times \frac{22}{7} \times 3.5 \times 3.5 \ m^2 = 154 \ m^2.$$

**Example 11.3.4.** A hemisphere dome of a building needs to be painted (see Fig. 11.3.3). If circumference of the base of the dome is 17.6 m, find the cost of painting it, given the cost of painting is 5 rs per 100 $cm^2$.

**Solution :** Since only the rounded surface of the dome is to be painted, we would need to find the curved surface area of the hemisphere to know the extent of painting that needs to be done. Now, circunference of the dome = 17.6 m. Therefore, $2\pi r = 17.6$

So, the radius of the dome $= 17.6 \times \frac{7}{2\times 22} = 2.8\ m$

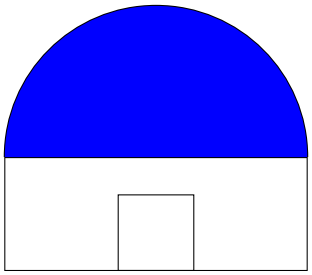


Figure 11.3.3: Building

The curved surface area of the dome $= 2\pi r^2$

$$= 2 \times \frac{22}{7} \times 2.8 \times 2.8\ m^2 = 49.28\ m^2.$$

Now, cost of painting 100 $cm^2$ is 5 rs. So, cost of painting 1 $m^2 = 10000\ cm^2 =$ 500 $r$.

Therfore, cost of painting the whole dome

$$= 500 \times 49.28 = 24640\ rs.$$

## 11.4 Solution of Exercise 11.2

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

**Problem 11.4.1.** Find the surface area of a sphere of radius:

$(i)$ 10.5 $cm$ $(ii)$ 5.6 $cm$ $(iii)$ 14 $cm$

**Solution :** (i) The radius of sphere is 10.5 cm.

The surface area of a sphere with radius 10.5 cm

$$4\pi r^2 = 4 \times \frac{22}{7} \times 10.5 \times 10.5 \ cm^2$$

$$= 1386 \ cm^2$$

(ii) The radius of sphere is 5.6 cm.

The surface area of a sphere with radius 5.6 cm

$$4\pi r^2 = 4 \times \frac{22}{7} \times 5.6 \times 5.6 \ cm^2$$

$$= 394.24 \ cm^2$$

(iii) The radius of sphere is 14 cm.

The surface area of a sphere with radius 14 cm

$$4\pi r^2 = 4 \times \frac{22}{7} \times 14 \times 14 \ cm^2$$

$$= 2464 \ cm^2$$

**Problem 11.4.2.** Find the surface area of a sphere of diameter :

(i) 14 cm (ii) 21 cm (iii) 3.5 m .

**Solution :** (i) The radius of a sphere $= \frac{Diameter}{2} = \frac{14}{2} = 7\ cm.$

The surface area of the sphere

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7\ cm^2 = 616\ cm^2$$

(ii) The radius of a sphere $= \frac{Diameter}{2} = \frac{21}{2} = 10.5\ cm.$

The surface area of the sphere

$$4\pi r^2 = 4 \times \frac{22}{7} \times 10.5 \times 10.5\ cm^2 = 1386\ cm^2$$

. (iii) The radius of a sphere $= \frac{Diameter}{2} = \frac{3.5}{2} = 1.75\ m.$

The surface area of the sphere

$$4\pi r^2 = 4 \times \frac{22}{7} \times 1.75 \times 1.75\ m^2 = 38.5\ m^2$$

**Problem 11.4.3.** Find total surface area of a hemisphere of radius 10.5 cm. (Use $\pi = 3.14$)

**Solution :** The radius of the sphere is 10.5 cm.

The total surface area of hemisphere

$$3\pi r^2 = 3 \times 3.14 \times 10.5^2\ cm^2 = 1038.55\ cm^2.$$

**Problem 11.4.4.** The radius of a spherical balloon increases from 7 cm to 14 cm as air being dumped into it. Find the ratio of surface area of the ballon in the two cases.

**Solution :** In frist case the radius of ballon is 7 cm. Then, the surface area of balloon

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \ cm^2 = 616 \ cm^2.$$

In case second the radius of spherical ballon is 14 cm. Then, the surface area of balloon

$$4\pi r^2 = 4 \times \frac{22}{7} \times 14 \times 14 \ cm^2 = 2464 \ cm^2.$$

The ratio of surface area of frist balloon to second balloon

$$= \frac{616}{2464} = \frac{1}{4}.$$

**Problem 11.4.5.** A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin-plating it on the inside at the rate of 16 rs per 100 $cm^2$.

**Solution :** The radius of hemispherical bowl is 10.5 cm. The curved surface area of hemispherical bowl

$$2\pi r^2 = 2 \times \frac{22}{7} \times 5.25 \times 5.25 \ cm^2 = 174.24 \ cm^2$$

Rate of tin-plating of bowl is 16 rs per 100 $cm^2$. So cost of tin-plating of the bowl

$$= 174 \times .16 = 27.87 \ rs.$$

**Problem 11.4.6.** Find the radius of a sphere whose surface area is 154 $cm^2$.

**Solution :** Let the radius of sphere is $r$ cm. Then, surface area of the sphere being $4\pi r^2$. According to question

$$4\pi r^2 = 154 \implies r^2 = \frac{7}{4 \times 22} \times 154 cm^2$$
$$r = \sqrt{\frac{49}{4}} = \frac{7}{2} = 3.5 \ cm.$$

**Problem 11.4.7.** The diameter of the moon is aproximately one fourth of the diameter of the earth. Find the ratio of their surface area.

**Solution :** Suppose the radius of the earth is $r_e$, then radius of moon is $r_m = \frac{1}{4} r_e$.

The surface area of moon

$$= 4\pi r_m^2 = 4\pi \left(\frac{1}{4} r_e\right)^2 = \frac{1}{4}\pi r_e^2$$

The surface area of earth

$$= 4\pi r_e^2$$

The ratio of surface area of moon and earth

$$= \frac{\frac{1}{4}\pi r_e^2}{4\pi r_e^2} = \frac{1}{16}.$$

**Problem 11.4.8.** A hemispherical bowl is made of steel, 0.25 $cm$ thick. The inner radius of bowl is 5 $cm$. Find the outer curved surface area of bowl.

**Solution :** The inner radius of steel bowl and thickness are 5 $cm$ and .25 $cm$ respectively. So, outer radius of steel bowl will be 5.25 $cm$.

The curved surface are of hemisphericl bowl will be

$$2\pi r^2 = 2 \times \frac{22}{7} \times 5.25 \times 5.25 \ cm^2 = 173.25 \ cm^2.$$

**Problem 11.4.9.** A right circular cylinder just encloses a sphere of radius $r$ (see Fig. 14.4.1 ). Find

i surface area of the sphere,

ii curved surface area of the cylinder,

iii ratio of the area obtained in (i) and (ii).

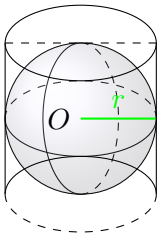



Figure 11.4.1: Sphere in cylinder

**Solution :** From Fig. (11.4.1), the radius of sphere is equal to radius of cylinder and equal to $r$.

(i) The surface area of sphere

$$= 4\pi r^2.$$

(ii) The height of cylinder is diameter of sphere and equal to $2r$.

The curved surface area of cylinder

$$= 2\pi rh = 2\pi r \times 2r = 4\pi r^2.$$

(iii) The ratio of area of sphere and cylinder $= \frac{4\pi r^2}{4\pi r^2} = 1 : 1$

## 11.5 Volume of a Right Circular Cone

Soppose the radius and height of the right circular cone are $r$ and $h$ respectively.

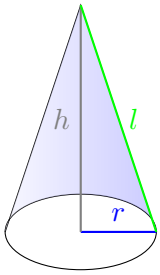



Figure 11.5.1: Cone

Volume of a Cone $= \frac{1}{3}\pi r^2 h$

where $r$ is the radius and $h$ is the height of the cone.

**Example 11.5.1.** The height and the slant height of a cone are 21 cm and 28 cm respectively. Find the volume of the cone.

**Solution :**

We know that $l^2 = r^2 + h^2$, so, $r = \sqrt{l^2 - h^2}$. Therefore $r = \sqrt{28^2 - 21^2} = 7\sqrt{7}\ cm$.

So, volume of the cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7\sqrt{7} \times 7\sqrt{7} \times 21\ cm^3$

$$= 7546\ cm^3.$$

**Example 11.5.2.** Monica has a pice of canvas whose area is 551 $m^2$. She use it to have a conical tent made, with a base radius of 7 $m$. Assuming that all the stiching margins and the wastage incured while cutting, amounts to aproximately 1 $m^2$, find the volume of the tent that can made with it.

**Solution :** Since the area of the canvas is 551 $m^2$ and area of the canvas wastage = 1 $m^2$, so net area of canvas available for making the conical tent $= (551 - 1)\ m^2 = 550\ m^2$.

Suppose the slant height of tent is $l$. The curved surface area and radius of the tent are 550 $m^2$ and 7 $m$ respectively. Therfore,

$$\pi r l = 550$$

$$\implies \frac{22}{7} \times 7 \times l = 550 \implies l = 25\ m.$$

Now, $l^2 = r^2 + h^2$, so, $h = \sqrt{l^2 - r^2}$. Therefore

$$h = \sqrt{25^2 - 7^2} = \sqrt{625 - 49}\ m = 24\ m.$$

So, volume of the conical tent

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24 \ m^3 = 1232 \ m^3.$$

## 11.6 Solution of Exercise 11.3

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

**Problem 11.6.1.** Find the volume of the right circular cone with

(i) radius 6 $cm$ , height 7 $cm$ (ii) radius 3.5 $cm$, and height 12 $cm$.

**Solution :** (i) The height $h = 7$ $cm$ and radius $r = 6$ $cm$ are of the right circular cone. The volume of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 6 \times 6 \times 7 = 264 \ cm^3.$$

(ii) (i) The height $h = 12$ $cm$ and radius $r = 3.5$ $cm$ are of the right circular cone. The volume of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 12 = 154 \ cm^3.$$

**Problem 11.6.2.** Find the capacity in liters of a conical vessel with

(i) radius 7 $cm$, slant height 25 $cm$, (ii) radius 12 $cm$, slant height 13 $cm$.

**Solution :** (i) The radius and slant height of the cone are 7 $cm$ and 25 $cm$ respectively. We know that, $h = \sqrt{l^2 - r^2}$.

So

$$h = \sqrt{25^2 - 7^2} = \sqrt{576} = 24 \ cm,$$

The volume of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24 \ cm^3 = 1232 \ cm^3.$$

The capacity of conical vessel $= \frac{1232}{1000} = 1.232 \ l.$

(i) The radius and slant height of the cone are 12 $cm$ and 13 $cm$ respectively. We know that, $r = \sqrt{l^2 - h^2}$.

So

$$r = \sqrt{13^2 - 12^2} = \sqrt{25} = 5 \ cm,$$

The volume of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 5 \times 5 \times 12 \ cm^3 = 314 \ cm^3.$$

The capacity of conical vessel $= \frac{314}{1000} = 3.14 \ l.$

**Problem 11.6.3.** The height of a cone is 15 $cm$. If its volume is 1570 $cm^3$, find the radius of the base (Use $\pi = 3.14$).

**Solution :** The volume and height of cone are 1570 $cm$ and 12 $cm$. Suppose the radius of base of cone is $r$ then, volume of cone $= \frac{1}{3}\pi r^2 h$. According to question

$$\frac{1}{3} \times 3.14 \times r^2 \times 15 = 1570 \implies r^2 = \frac{1570 \times 3}{3.14 \times 15} = 100$$

$$r = 10\ cm$$

Hence radius of base of cone is 10 $cm$.

**Problem 11.6.4.** If the volume of a right circular cone of height 9 $cm$ is $48\pi\ cm^3$, find the diameter of its base.

**Solution :** The volume and height of cone are $48\pi\ cm$ and 9 $cm$. Suppose the radius of base of cone is $r$ then, volume of cone $= \frac{1}{3}\pi r^2 h$. According to question

$$\frac{1}{3} \times \pi \times r^2 \times 9 = 48\pi \implies r^2 = \frac{48 \times \pi}{3 \times \pi} = 16$$

$$r = 4\ cm$$

Hence diameter of base of cone is $d = 2r$ or 8 $cm$.

**Problem 11.6.5.** A conical pit of top diameter 3.5 $m$ is 12 $m$ deep. What is its capacity in kiloliters ?

**Solution :** The shape of pit is conical. The diameter and deep of conical pit are 3.5 $m$ and 12 $m$ respectively. So, radius of top of pit being $\frac{3.5}{2} = 1.75\ m$.

Volume of pit $= \frac{1}{3} \times \pi \times r^2 \times h$

$$= \frac{1}{3} \times \frac{22}{7} \times 1.75 \times 1.75 \times 12\ m^3 = 38.5\ m^3$$

Hence capacity of pit 38.5 kiloliter.

**Problem 11.6.6.** The volume of right circular cone is 9856 $cm^3$. If the diameter of the base is 28 $cm$, find

i height of cone

ii slant height of the cone

iii curved surface area of the cone.

**Solution :** The diameter of the cone is 28 $cm$, so radius of cone be 14 $cm$.

(i) Let height of the cone is $h$. so, volune of the cone $= \frac{1}{3}\pi r^2 h$. According to question,

$$\frac{1}{3} \times \frac{22}{7} \times 14 \times 14 \times h = 9856$$

$$h = 9856 \times \frac{3 \times 7}{22 \times 14 \times 14} = 48 \ cm$$

(ii) The radius and height of the cone are 14 $cm$ and 48 $cm$. Therefore slant height $l = \sqrt{r^2 + h^2}$. So

$$l = \sqrt{14^2 + 48^2} = \sqrt{2500} = 50 \ cm.$$

(iii) The curved surface area of cone with radius and slant height are 14 $cm$ and 48 $cm$. The curved surface area of the cone is $\pi rl$. So curved surface area of the cone

$$= \pi rl = \frac{22}{7} \times 14 \times 50 \ cm^3 = 2200 \ cm^3.$$

**Problem 11.6.7.** A right triangle ABC with sides 5 $cm$ 12 $cm$ and 13 $cm$ is resolved about the side 12 $cm$. Find the volume of the solid so obtained.

**Solution :** In triangle ABC BC is revolved about AC then the height of cone and radius of its base being $h = 12\ cm$ and $r = 5\ cm$.

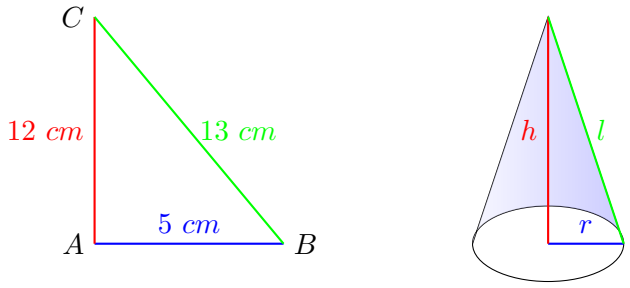



Figure 11.6.1: Triangle ABC convertion in cone after $BC$ revolving about $AC = 12\ cm$

So volume of the cone being,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 5 \times 5 \times 12\ cm^3 = 100\pi\ cm^3.$$

**Problem 11.6.8.** If the triangle ABC in the question 7 above is revolved about the the side 5 $cm$, then find the volume of the solid so obtained. Find also the ratio of the volumes of the two solids obtained in Question 7 and 8.

**Solution :** In triangle ABC, BC is revolved about AB then the height of cone and radius of its base being $h = 5\ cm$ and $r = 12\ cm$.

So volume of the cone being,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 12 \times 12 \times 5\ cm^3 = 240\pi\ cm^3.$$

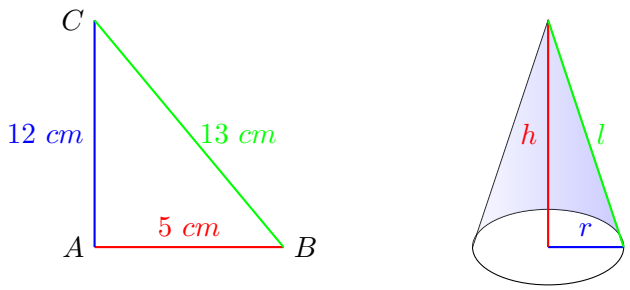



Figure 11.6.2: Triangle ABC convertion in cone after $BC$ revolving about $AB = 5\ cm$

Ratio of volume from Question 7 and 8 $= \frac{100\pi}{240\pi} = \frac{5}{12}$.

**Problem 11.6.9.** A heap of wheat is in the form of a cone whose diameter is 10.5 $m$ and height is 3 $m$. Find its volume. The heap is to be covered by canvas to protect it from rain. Find the area of the canvas required.

**Solution :**

The diameter and height of heap of the wheat are 10.5 $m$ and 3 $m$ respectively. So radius of heap being 5.25 $m$. The volume of heap $= \frac{1}{3}\pi r^2 h$.

$$= \frac{1}{3} \times \frac{22}{7} \times 5.25 \times 5.25 \times 3\ m^3 = 86.625\ m^3$$

The slant height of conical heap $l = \sqrt{r^2 + h^2} = \sqrt{(5.25)^2 + 3^2} = 6.05\ m$.

The area of canvas to covered the heap = curved surface area of heap.

The curved surface area of heap $= \pi r l$

$$= \frac{22}{7} \times 5.25 \times 6.05\ m^2 = 99.825\ m^2$$

Hence required area of canvas = 99.825 $m^2$.

## 11.7 VOLUME OF A SPHERE

Let the radius of a sphere is $r$, then,

Volume of a Sphere = $\frac{4}{3}\pi r^3$,

Volume of a Hemisphere = $\frac{2}{3}\pi r^3$

where the $r$ is radius of a sphere.

**Example 11.7.1.** Find the volume of a sphere of radius 11.2 $cm$.

**Solution :** Required volume = $\frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 11.2 \times 11.2 \times 11.2\ m^3 = 5887.32\ cm^3.$$

**Example 11.7.2.** A shot-putt is a metallic sphere of radius 4.9 $cm$. If the density of the metel is 7.8 $g$ per $cm^2$, find the mass of the shot-putt.

**Solution :** The radius of the sphere = 4.9 $cm$. So the volume of the sphere = $\frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 4.9 \times 4.9 \times 4.9\ cm^3 = 493\ cm^3$$

The density of metal = 7.8 $g$ per $cm^3$. So, mass of shot-putt = $volume \times density$

$$= 493 \times 7.8\ g = 3.85\ kg.$$

**Example 11.7.3.** A hemisphere bowl has a radius of 3.5 $cm$. What would be the volume of water it would contain ?

**Solution :** The radius of hemispherical shape bowl 3.5 $cm$. So, volume of water

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5\ cm^3 = 89.8\ cm^3$$

So, volume of water would contain in bowl 89.8 $cm^3$.

## 11.8 Solution of Exericse 11.4

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

**Problem 11.8.1.** Find the volume of a sphere whose radius is

(i) 7 $cm$ (ii) 0.63 $m$

**Solution :** (i) The radius of the sphere is 7 $cm$. So, volume of sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 7 \times 7 \times 7\ cm^3 = 1437.34\ cm^3.$$

(ii) The radius of the sphere is 0.63 $m$. So, volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 0.63 \times 0.63 \times 0.63 \ m^3 = 1.05 \ m^3.$$

**Problem 11.8.2.** Find the amount of water displaced by a solid spherical ball of diameter

(i) 28 $cm$ (ii) 0.21 $m$.

**Solution :** (i) The amount of water displaced by the solid spherical ball is equal to volume of the solid sphere. The diameter of sphere = 28 $cm$. So, radius of the sphere = 14 $cm$.

The volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 14 \times 14 \times 14 \ cm^3 = 11498.67 \ cm^3.$$

Volume of 1000 $cm^3 = 1$ $l$, so, 11498.67 $cm^3 = \frac{11498.67}{1000}$ $l = 11.50$ $l$.

(ii) The amount of water displaced by the solid spherical ball is equal to volume of the solid sphere. The diameter of sphere = 0.21 $m$. So, radius of the sphere = 0.105 $m$.

The volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 0.105 \times 0.105 \times 0.105 \ m^3 = 0.004851 \ m^3.$$

Volume of the sphere $0.004851 \times 1000000 \ cm^3 = 4851 \ cm^3$ Volume of 1000 $cm^3 =$ 1 $l$, so, 4851 $cm^3 = \frac{4851}{1000}$ $l = 4.851$ $l$.

**Problem 11.8.3.** The diameter of a metallic ball is 4.2 $cm$. What is the mass of the ball, if the density of the metal is 8.9 $g$ per $cm^3$ ?

**Solution :** The diameter of the ball is 4.2 $cm$ so, radius of the ball being 2.1 $cm$. Hence Volume of metallic ball $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 2.1 \times 2.1 \times 2.1 \ cm^3 = 38.808 \ cm^3$$

The density of metal is 8.9 $g$. Mass of metallic ball $= Volume \times Density$

$$= 38.808 \times 8.9 \ g = 345.39 \ g.$$

**Problem 11.8.4.** The diameter of the moon is approximately one-fourth of the diameter of the earth . What fraction of the volume of the earth is volume of the moon.

**Solution :** Let the diameter of earth is $2r$. So diameter of the moon being $\frac{r}{2}$. Now, the radius of the earth and moon will be $r$ and $\frac{r}{4}$ respectively.

The volume of the earth $= \frac{4}{3}\pi r^3$ and volume of the moon $= \frac{4}{3}\pi\frac{r^3}{64}$.

The ratio of volume of earth and moon $= \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi\frac{r^3}{64}} = 64 : 1.$

Hence volume of the moon is one-sixtyfourth of the volume the earth.

**Problem 11.8.5.** Find the volume of a sphere whose surface area is 154 $cm^2$.

**Solution :** Let the radius of a sphere is $r$. So, the surface area of the sphere $= 4\pi r^2$. According to question

$$4\pi r^2 = 154 \implies r^2 = \frac{7}{22 \times 4} \times 154 = \frac{49}{4} \ cm$$

$$r = \frac{7}{2}\ cm$$

Now the volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi\left(\frac{7}{2}\right)^3\ cm^3 = 57.17\ cm^3.$$

**Problem 11.8.6.** How many litres of milk can be hemispherical bowl of diameter 10.5 $cm$ hold ?

**Solution :** The diameter of hemispherical bowl is 10.5 $cm$ then radius of hemispherical bowl 5.25 $cm$. So, volume of hemispherical bowl $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times 5.25 \times 5.25 \times 5.25\ cm^3 = 300.3\ cm^3$$

Water holding capacity of bowl is 300.3 $ml$.

**Problem 11.8.7.** A hemispherical tank is made up of an iorn sheet 1 $cm$ thick. if the inner radius is 1 $m$, then find the volume of the iron used to make the tank.

**Solution :** The inner and outer radius of hemispherical tank are 1 $m$ and 1.01 $m$.

The volume of tank with inner radius $= \frac{2}{3} \times \frac{22}{7} \times 1 \times 1 \times 1\ m^3 = 2.09523809\ m^3$

The volume of tank with uoter radius $= \frac{2}{3} \times \frac{22}{7} \times 1.01 \times 1.01 \times 1.01\ m^3 = 2.15872590\ m^3$

The volume of iron $= (2.15872590 - 2.09523809)\ m^3 = 0.06348781\ m^3$

**Problem 11.8.8.** A dome of a building is in the form of a hemisphere. From inside, it was white-washed at the cost of 4989.60 $rs$. If the cost of white-washing is 20 $rs$ per square meter, Find the

(i) inside surface area of the dome (ii) volume of the air inside the dome.

**Solution :** (i) The rate of white-washing of dome is 20 rs per square meter and cost of white-washing is 4989.60 rs.

So, the inside surface area of dome $= \frac{4989.60}{20}\ m^2 = 249.48\ m^2$

(ii) Let the radius of dome is $r$ then surface area of dome $2\pi r^2$.

According to question

$$2 \times \frac{22}{7} r^2 = 249.48 \implies r^2 = 39.69$$

$$r = \sqrt{39.69} = 6.3\ m.$$

The volume of the air inside dome $= \frac{2}{3}\pi r^3$.

$$= \frac{2}{3} \times \frac{22}{7} \times 6.3 \times 6.3 \times 6.3\ m^3 = 523.908\ m^3$$

**Problem 11.8.9.** Twenty seven solid iron sphere, each of radius $r$ and surface area S are melted to form a sphere with surface area S'. Find the

(i) radius $r'$ of the new sphere (ii) ratio of S and S'.

**Solution ;** (i) The volume a solid iron sphere $= \frac{4}{3}\pi r^3$ so, volume of 27 solid iron sphere $27 \times \frac{4}{3}\pi r^3 = 36\pi r^3$. Let radius of new sphere is $\acute{r}$ then

$$36\pi r^3 = \frac{4}{3}\pi \acute{r}^3 \implies \acute{r} = 3r$$

(ii) The surface area of the small solid iron sphere $S = 4\pi r^2$

The surface area of the big solid iron sphere $\acute{S} = 4\pi \acute{r}^2 = 36\pi r^2$

The ratio of surface area of small and big solid iron sphere $= \frac{4\pi r^2}{36\pi r^2} = \frac{1}{9}$.

**Problem 11.8.10.** A capsule of medicien is in the shape of a sphere of diameter 3.5 $mm$. How much medicien (in $mm^3$) is needed to fill this capsule.

**Solution :** The diameter of spherical capsule is 3.5 $mm$ so, radius be 1.75 $mm$.

Now, the volume of capsule $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 1.75 \times 1.75 \times 1.75 \ mm^3$$

$$= 22.46 \ mm^3.$$

Therefore the volume of medicien to fill capsule $= 22.46 \ mm^3$.

# 12 STATISTIC

## 12.1 Introduction

In earlyer class we have been studied the representation of data by tables. Now in this chapter we studies the represntation data by graphical method. There will be discuss three technique (i) bar graphs (ii) histograms of uniform width, and of varying width (iii) Frequency polygons.

**Bar Graphs :** A bar graph is a pictorial representation of data in which usually bars of uniformsly width are drawn with equal spacing between them on one axis (say the x-axis), depicting the variable. The values of the variable are shown on the other axis (say the y-axis) and height of the bar depends on the values of the variables.

**Example 12.1.1.** In a particular section of class $IX$, 40 students were asked about the month of their birth and the following graph was prepared for the data so obtained :

Observed the bar graph below and answer the followinf questions:

i How many students were born in the month of November.

ii In which month were the maximum number of students born.

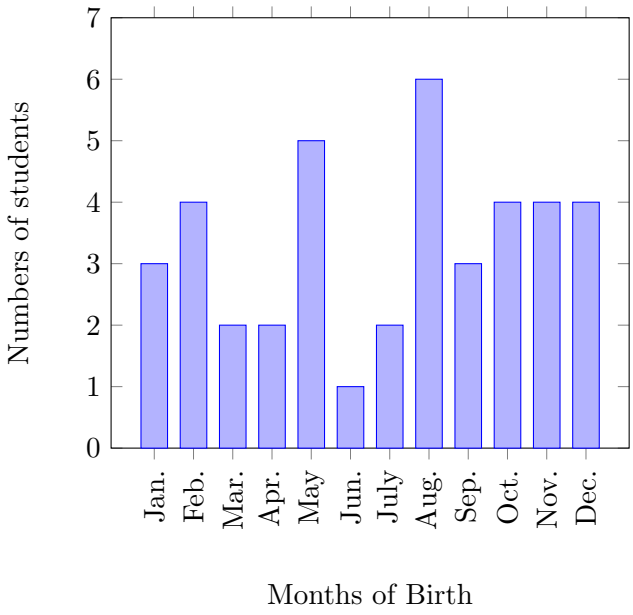



Figure 12.1.1

**Solution :** Note that the variable here is the "month of birth" and the value of the variable is the 'Number of student born'

i 4 students were born in the month of November.

ii The maximum number of students were born in the month of August.

**Example 12.1.2.** A family with a monthaly income of 20000 rs had planned the following expenditure per month under various haeds:

Draw the bar graph for the data given in Table 1.

**Solution :** We draw the bar graph of this data in the following steps. Note that unit in the second column is thousand rupees. So, '4' against 'grocery'

| Heads | Expenditure (in thousand rupees) |
|---|---|
| Grocery | 4 |
| Rent | 5 |
| Education of children | 5 |
| Medicien | 2 |
| Fuel | 2 |
| Entertainment | 1 |
| Miscellaneous | 1 |

Table 1

means 4000 rs.

1. We represent the heads (variable) on horizontal axis choosing any scale, since the width of the bar is not important. But for clarity, we take equal width for all bars and maintain equal gaps in btween. Let one head be represent by one unit.

2. We represent the expenditure (value) on the vertical axis. Since the maximum expenditure is 5000 rs, we can choose the scale as 1 $unit = 1000$ rs.

3. To represent our frist head, i.e., grocery, we draw a rectangular bar with width 1 unit and height 4 unit.

4. Similarly, other Heads are represented leaving a gap of 1 unit in btween two consecutive bars.

The graph is drawn in fig. (12.1.2).

Here, you can easily visualise the relative characteristics of the data at a glance.

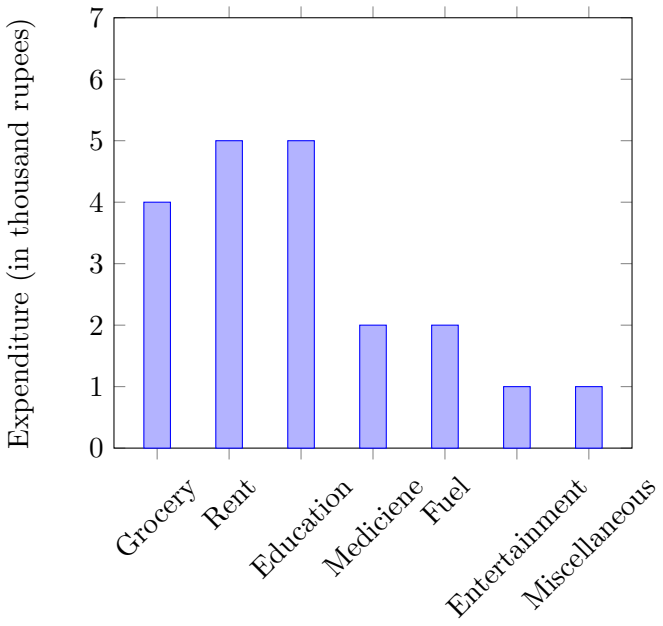



Figure 12.1.2

**Histogram :** This is a form of representation like the bar graph, but it is used for continuous class intervals. For instance, consider the frequency distribution Tabel 2, representing the weight of 36 students of a class:

| Weight (in kg) | Number of students |
|---|---|
| 30.5 - 35.5 | 9 |
| 35.5 - 40.5 | 6 |
| 40.5 - 45.5 | 15 |
| 45.5 - 50.5 | 3 |
| 50.5 - 55.5 | 1 |
| 55.5 - 60.5 | 2 |
| Total | 36 |

Table 2

Let us represent the data given in Table-2 graphically as follows:

i We represent the weight on the horizontal axis on a sutable scale. We can choose the scale as 1 $cm = 5\ kg$. Also, since the first class interval is starting from 30.5 and not zero, so, we take 0 equivalent to 30.5 on the axis.

ii We represent the number of students (frequency) on the vertical axis on a suitable scale. Since the maximum frequency is 15, we need to choose the scale to accomodate this maximum frequncy.

iii We now draw rectangles (or rectangular bars) of width equal to the class-size and lengths according to the frequency of the corresponding class intervals.

iv In this way, we obtain the graph as shown in Fig. (12.1.3).

Observe that since there are no gap in between consecutive rectangles, the resultant graph appears like solid figure. This is called a *histogram*, which is a graphical representation of a grouped frequency distrubution with continuous classes.

**Example 12.1.3.** A teacher wanted to analyse the performance of two sections of students in a mathematics test of 100 marks. Looking at their performance, she found that a few students got under 20 marks and few got 70

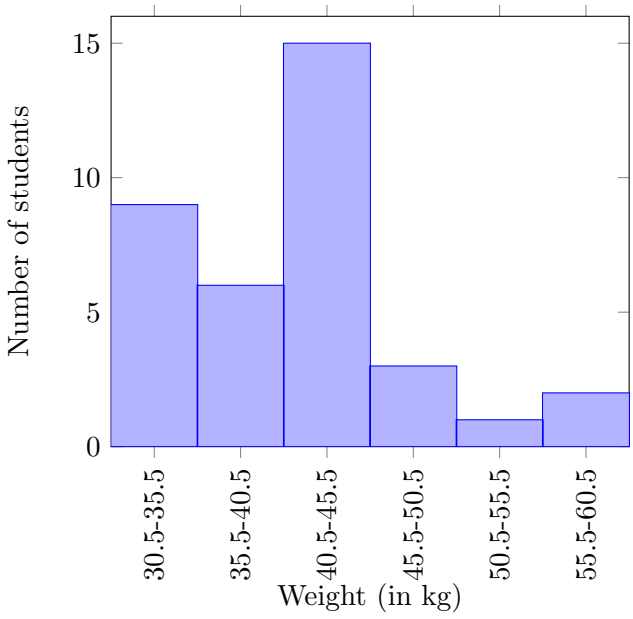



Figure 12.1.3

marks or above. So, she decided to group them into intervals of varying size as follows: $0-20,\ 20-30,\ \ldots 60-70,\ 70-100$. Then she formed the following table:

| Marks | Number of students |
|---|---|
| 0 - 20 | 7 |
| 20 - 30 | 10 |
| 30 - 40 | 10 |
| 40 - 50 | 20 |
| 50- - 60 | 20 |
| 60 - 70 | 15 |
| 70 - above | 8 |
| Total | 90 |

Table 3

A histogram for this table was prepared by a student as shown in Fig. (12.1.4).

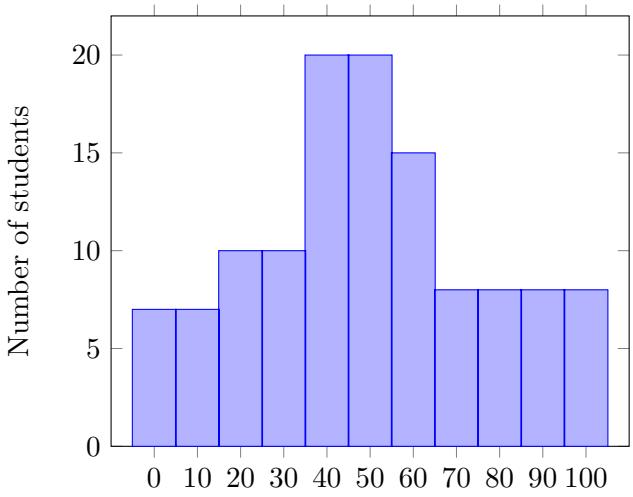



Figure 12.1.4

Carefully analyse this graphical representation. Do you think that it correctly represents the data? No, the graph is giving us a misleading picture. As we have mentioned earlier, the areas of the rectangles are proportional to the frequencies in a histogram. Earlier this problem did not arise, because the widths of all the rectangles were equal. But here, since the widths of the rectangles are varying, the histogram (12.1.4) does not give correct picture. So, we need to make certain modifications in the lengths of rectangles so that the areas are again proportional to the frequency.

The steps to be followed are as given below:

1. Select a class interval with the minimum class size. In the eaxample above, the minimum class-size is 10.

2. The lengths of the rectangles are then modified to be proportionate to the class-size 10.

| Marks | Frequency | Width of the class | Length of the rectangle |
|---|---|---|---|
| 0 - 20 | 7 | 20 | $\frac{7}{20} \times 10 = 3.5$ |
| 20 - 30 | 10 | 10 | $\frac{10}{10} \times 10 = 10$ |
| 30 - 40 | 10 | 10 | $\frac{10}{10} \times 10 = 10$ |
| 40 - 50 | 20 | 10 | $\frac{20}{10} \times 10 = 20$ |
| 50- - 60 | 20 | 10 | $\frac{20}{10} \times 10 = 20$ |
| 60 - 70 | 15 | 10 | $\frac{15}{10} \times 10 = 15$ |
| 70 - above | 8 | 30 | $\frac{8}{30} \times 10 = 2.67$ |

Table 4

Since we have calculated these lengths for an interval of 10 marks in each case, we may call these lengths as " proportion of students per 10 marks interval".

So, the correct histogram with varying width is given by Fig. (12.1.5).

**Frequency Polygon :** There is yet another visual way of representing quantitive data and its frequncies. This is a *Polygon.* In the histogram, when class interval represented by mid point of class and after joining these mid points, they form a polygon that is *frequency polygon.*

Now, for Fig. (12.1.6) and (12.1.7). The frequency polygon given as Fig.

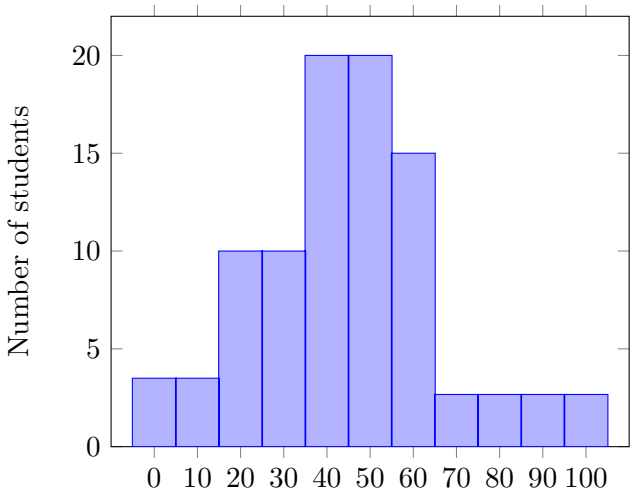



Figure 12.1.5

(12.1.7).

Although, there exist no class preceding the lowest class and no class succeeding the highest class, addition of the two class interval with zero frequency enables us to make the area of the frequency polygon the same as the area of the histogram.

**Example 12.1.4.** Consider the marks, out of 100, obtained by 51 students of a class in a test, given in Table 5.

Draw a frequency polygon corresponding to this frequency distribution table.

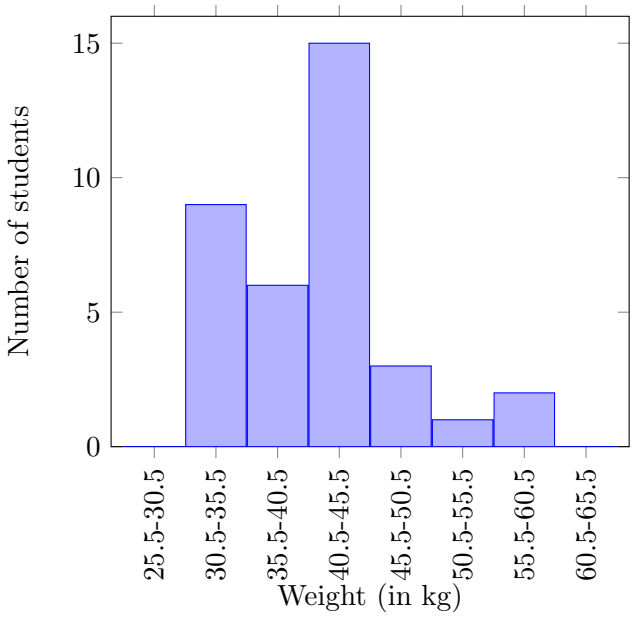



Figure 12.1.6

**Solution :** Let us draw the frequency polygon without drawing histogram. We mark mid-point of class-interval as the mid-point class-interval $0-10$ is 5 and its frequency 5, similarly draw mid-points of each class-interval then join these points which give the frequency polygon.

**Note :** The mid-point of the class-class interval is called *class-mark.*

$$\text{Class-mark} = \frac{\textit{Upper limit}+\textit{Lower limit}}{2}$$

**Example 12.1.5.** In a city, the weekly observations made in a study on the

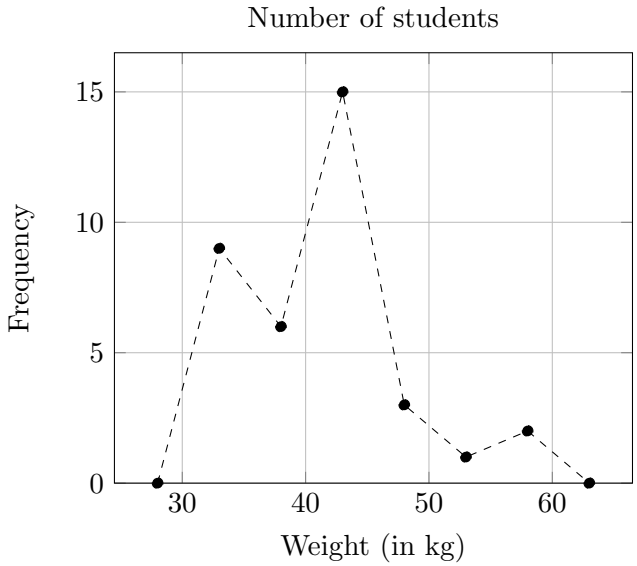



Figure 12.1.7

| Marks | Frequency |
|---|---|
| 0 - 10 | 7 |
| 10 - 20 | 10 |
| 20 - 30 | 4 |
| 30 - 40 | 6 |
| 40 - 50 | 7 |
| 50 - 60 | 3 |
| 60 - 70 | 2 |
| 70 - 80 | 2 |
| 80 - 90 | 3 |
| 90 - 100 | 9 |
| Total | 51 |

Table 5

cost of living index are given in the following table-6 :

Draw a frequency polygon for the data above (without constructing a histogram).

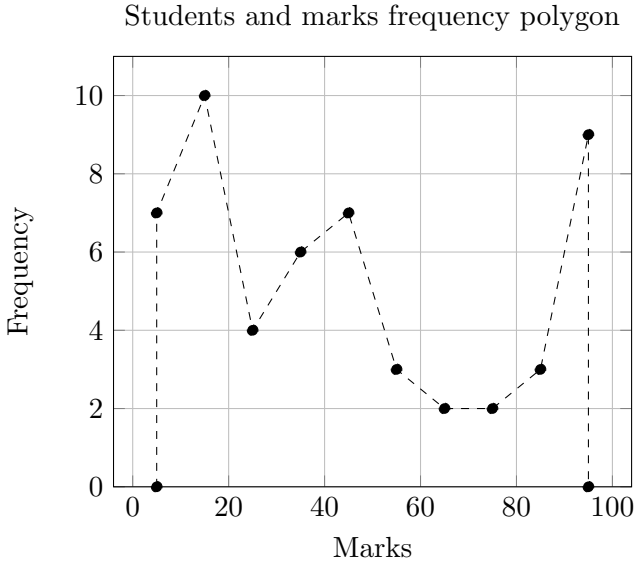



Figure 12.1.8

| Cost of living index | Number of week |
|---|---|
| 140 - 150 | 5 |
| 150 - 160 | 10 |
| 160 - 1700 | 20 |
| 170 - 180 | 9 |
| 180 - 190 | 6 |
| 190 - 200 | 2 |
| Total | 52 |

Table 6

**Solution :** since we want to draw a frequency polygon without a histogram, let us find the class-marks of the classes given above, that is $140 - 150,\ 150 - 160\ \ldots.$

For $140 - 150$, the upper limit $= 150$, and lower limit $= 140$.

So, the class-mark $= \frac{150+140}{2} = \frac{290}{2} = 145$.

Similarly, we find the class-marks of the other classes as well. So, new table cunstructed as:

| Cost of living index | Number of week | Class-mark |
|---|---|---|
| 140 - 150 | 5 | 145 |
| 150 - 160 | 10 | 155 |
| 160 - 1700 | 20 | 165 |
| 170 - 180 | 9 | 175 |
| 180 - 190 | 6 | 185 |
| 190 - 200 | 2 | 195 |
| Total | 52 | |

Table 7

Now from table-7 we can draw a frequency polygon by plotting the classs-marks along the horizontal axis, the frequency along the vertical -axis and then plotting and joininng the points (145, 5), (155, 10), (165, 20), (175, 9), (185, 6), (195, 2) by line segments.

**Use of frequency polygon :** The frequency polygon are used when the data is continuous and very large. It is very useful for compairing two different sets of data of the same nature.

## 12.2 Solution of Exercise 12.1

**Problem 12.2.1.** A survey conducted by an organisation for the cause of illness and death among the women between the ages $15-44$ (in years) world-wide, found the following figure (in %) :

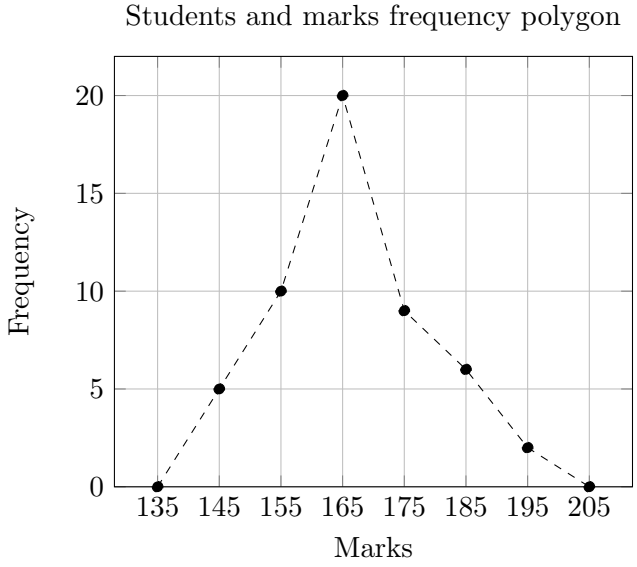



Figure 12.1.9

| S.No. | Causes | Female fatality rate (%) |
|---|---|---|
| 1. | Reproductive health condition | 31.8 |
| 2. | Neuropsychiatric conditions | 25.4 |
| 3. | Injuries | 12.4 |
| 4. | Cardiovascular conditions | 4.3 |
| 5. | Respiratory conditions | 4.1 |
| 6. | Other causes | 22.0 |

Table 8

i Represent the information given table-8 graphically.

ii Which condition is the major cause of women's ill health and death worldwide?

iii Try to find out, with the help of your teacher, any two factors which play a major role in the cause in (ii) above being the major cause.

**Solution :** There are two variable in given data cause and death rate in (%). Now we assign the cause along horizontal axis and rate of death along vertical axis and width of each bars is equal and takimg suotable scale.

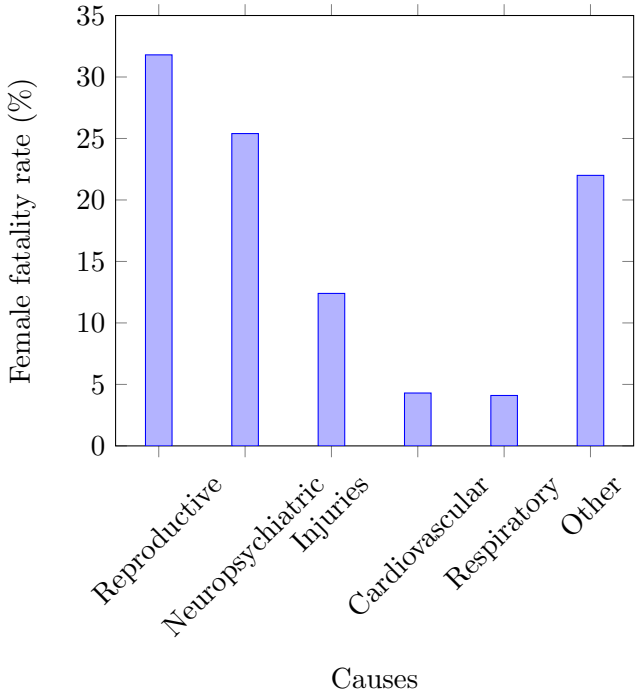



Figure 12.2.1

i The graphival representation of given information is represented by Fig. (12.2.1).

ii According Fig. (12.2.1), the longest bar is reproductive health conditions so the reproductivie health condition is the major cause of women's ill health and death worldwide.

iii The injuries and other cause play major factor in (ii) being major cause.

**Problem 12.2.2.** The following data on the number of girls (to the nearest ten) per thousand boys in different section of Indian society is given below.

| Section | Number of girls per thousand boys |
|---|---|
| Scheduled Caste (SC) | 940 |
| Scheduled Tribe (ST) | 970 |
| Non SC/ST | 920 |
| Backward districts | 950 |
| Non-backwarddistricts | 920 |
| Rular | 930 |
| Urban | 910 |

Table 9

i Represent the information above by a bar graph.

ii In the classroom discuss what conclusions can arrived at from the graph.

**Solution :** Now we take section along horizontal axis Numbers of girls along vertical axis as suitable scales.

i The given information in question is represented by Fig. (12.2.2).

ii From Fig.(12.2.2), Longest and shortest length of bars are Scheduled cast and Urban section of India society in the classroom. So, numbers of girls in Scheduled cast is hieghest and in urban is lowest in section.

**Problem 12.2.3.** Given below are the seats won by different political parties in the polling outcome of a state assembly elections:

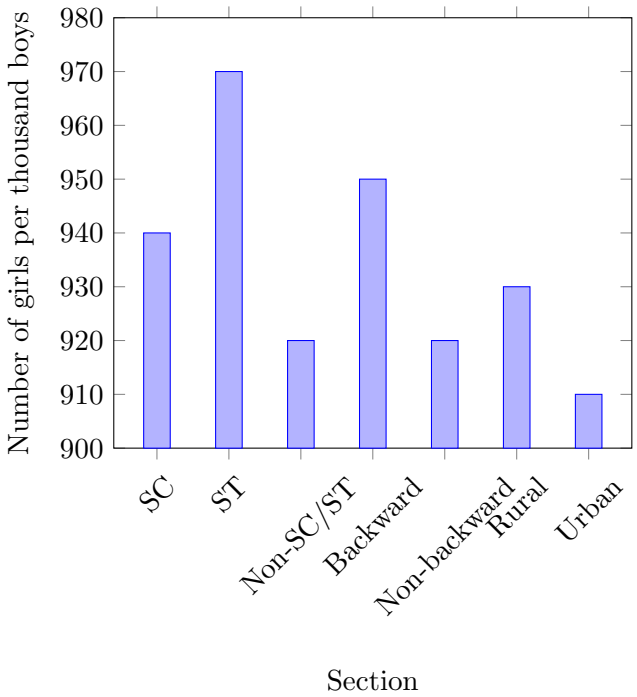



Figure 12.2.2

| Political Party | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Seat won | 75 | 55 | 37 | 29 | 10 | 37 |

Table 10

i Draw the bar graph to represent the polling results.

ii Which political party won the maximum number of seats?

**Solution :** Now we take political party along horizontal axis situated equal distance and numbers of seat won by political party along vertical axis for suitable scale.

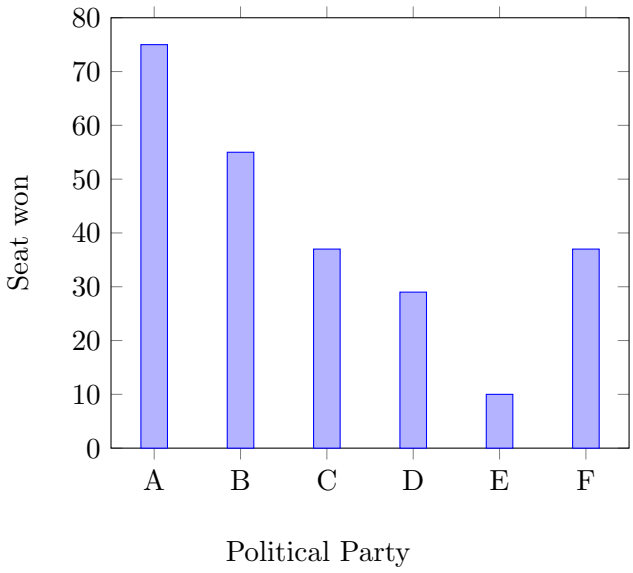



Figure 12.2.3

i The bar graph of the given information is represent by Fig. (12.2.3).

ii In the Fig. (12.2.3), the height of political party A is maximum, so the political party A won maximum number of seats.

**Problem 12.2.4.** The length of 40 leaves of a plant are measured correct to one milimetere and the obtained data is represented in the following table-11:

i Draw a histogram to represent the given data. [Hint: Frist make the class intervals continuous]

ii Is there any other suitable graphical representation for the same data?

| Length (in mm) | Number of leaves |
|---|---|
| 118-126 | 3 |
| 127-135 | 5 |
| 136-144 | 9 |
| 145-153 | 12 |
| 154-162 | 5 |
| 163-172 | 4 |
| 172-180 | 2 |

Table 11

iii Is it correct to conclude that the maximum number of leaves are 153 $mm$ long? why?

**Solution :** The given data is not continuous, so we need to make class-interval continuous. The minimum class-size of data is 8 and there is difference of 1 in each class-interval. So taking class-size $8 + 1 = 9$ and modified apropriate proportion length of class-size 9.

| Length (in mm) | Number of leaves/frequency | Width of the class | Length of the rectangle |
|---|---|---|---|
| 117.5-126.5 | 3 | 9 | $\frac{3}{9} \times 8 = 2.67$ |
| 126.5-135.5 | 5 | 9 | $\frac{5}{9} \times 8 = 4.44$ |
| 135.5-144.5 | 9 | 9 | $\frac{9}{9} \times 8 = 8$ |
| 144.5-153.5 | 12 | 9 | $\frac{12}{9} \times 8 = 10.67$ |
| 153.5-162.5 | 5 | 9 | $\frac{5}{9} \times 8 = 4.44$ |
| 162.5-171.5 | 4 | 9 | $\frac{4}{9} \times 8 = 3.56$ |
| 171.5-180.5 | 2 | 9 | $\frac{2}{9} \times 8 = 1.78$ |
| Total | 40 | | |

Table 12

Now the new modified rectangle lenght of each class-interval is given in table-12.

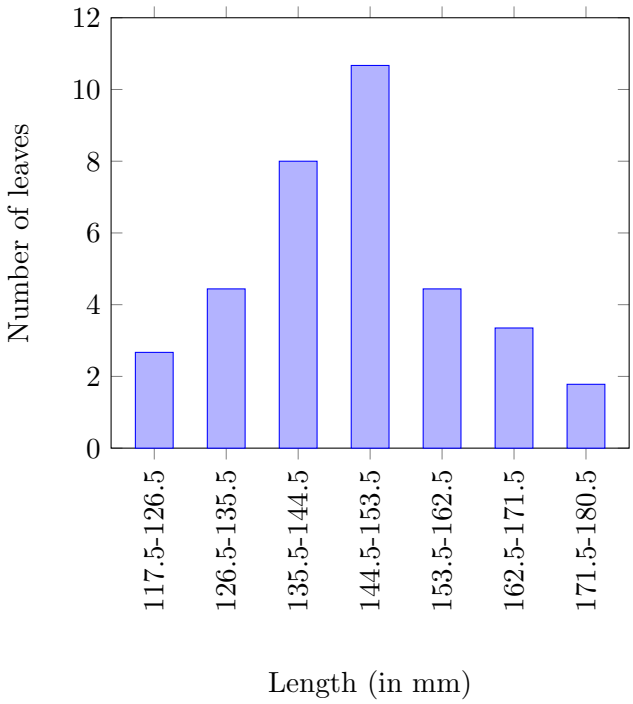



Figure 12.2.4

Taking class-interval along horizontal axis and modified rectangle length along vertical axis and draw the histogram.

i The histogram for given information in question is given in Fig.(12.2.4).

ii The frequency polygon is another graphical representation of same data.

iii No, it is not correct because the maximum number of leavs 153 $mm$ long are $= 8 \times 9 = 72$

**Problem 12.2.5.** The following table-13 the life time of 400 neon lamps:

| Life time (in hours) | Number of lamps |
|---|---|
| 300-400 | 14 |
| 400-500 | 56 |
| 500-600 | 60 |
| 600-700 | 86 |
| 700-800 | 74 |
| 800-900 | 62 |
| 900-1000 | 48 |

Table 13

i Represent the given information with the help of a histogram.

ii How many lamps have a life time of more than 700 hours?

**Solution:** The information in question with continuous class-intervals, so we can use data directly from question without modification.

Taking the class-interval along horizontal axis and life time of lamps along vertical axis.

i The histogram representation of question is given in Fig.(12.2.5).

ii The lamps which life time more than 700 hours belogs the interval $700-800,\ 800-900$ and $900-1000$.

The number of lamps belong in interval $700-800,\ 800-900$ and $900-1000$ is area of rectangle in histogram of these intervals.

The number of lamps, which life time more than 700 hours $= 74+62+48 = 184$ lamps.

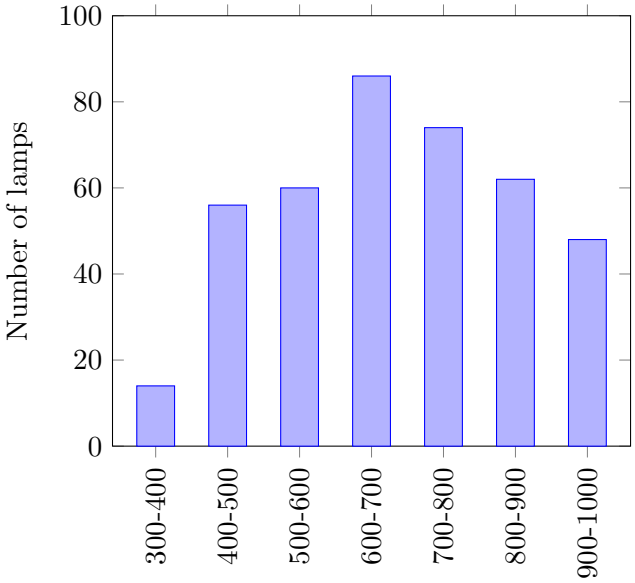



Figure 12.2.5

**Problem 12.2.6.** The following table-14 gives the distribution of two sections according to the marks obtained by them:

| Section A | | Section B | |
|---|---|---|---|
| Marks | Frequency | Marks | Frequency |
| 0-10 | 3 | 0-10 | 5 |
| 10-20 | 9 | 10-20 | 19 |
| 20-30 | 17 | 20-30 | 15 |
| 30-40 | 12 | 30-40 | 10 |
| 40-50 | 9 | 40-50 | 1 |

Table 14

Represent the marks of the students of both the sections on the same graph

by two frequency polygons. From the two polygon compare the performance of the two sections.

**Solution:** To draw frequency polygon we need class-mark of each class-interval. Class-mark $= \frac{Upper\ limit + Lower\ limit}{2}$. Now construct table with class-mark of given data.

| Section A | | | | Section B | |
|---|---|---|---|---|---|
| Marks | Class-mark | Frequency | Marks | Class-mark | Frequency |
| 0-10 | 5 | 3 | 0-10 | 5 | 5 |
| 10-20 | 15 | 9 | 10-20 | 15 | 19 |
| 20-30 | 25 | 17 | 20-30 | 25 | 15 |
| 30-40 | 35 | 12 | 30-40 | 35 | 10 |
| 40-50 | 45 | 9 | 40-50 | 45 | 1 |
| Total | | 50 | | 50 | |

Table 15

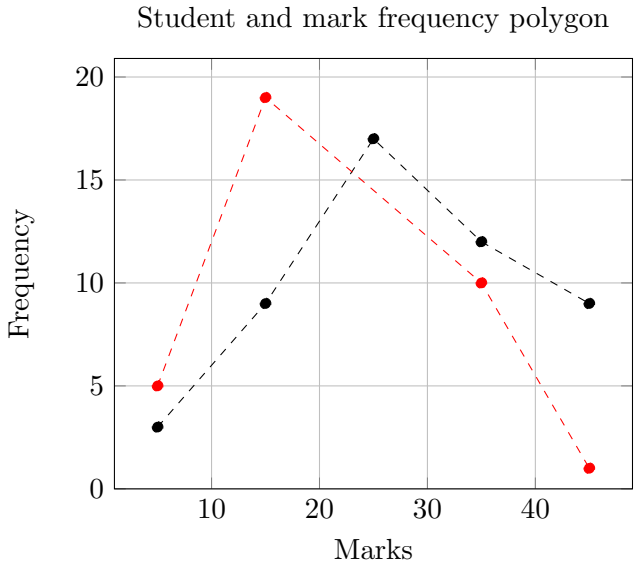



Figure 12.2.6

The frequency polygon of two section A and B drawn by red and black color. In frequency polygon Fig.(12.2.6), less than 22 mark section A frequency polygon below to frequency polygon of section B, but greater than 22 mark the frequency polygon of section A above the frequency polygon of section B. Hence it shows that section a student are more talented than section B.

**Problem 12.2.7.** The runs scored by two teams A and B on the first 60 balls in a cricket match are given in table-16:

| Number of balls | Team A | Team B |
|---|---|---|
| 1-6 | 2 | 5 |
| 7-12 | 1 | 6 |
| 13-18 | 8 | 2 |
| 19-24 | 9 | 10 |
| 25-30 | 4 | 5 |
| 31-36 | 5 | 6 |
| 37-42 | 6 | 3 |
| 43-48 | 10 | 4 |
| 49-54 | 6 | 8 |
| 55-60 | 2 | 10 |

Table 16

Represent the data of both the teams on the same graph by frequency polygons.

**Solution:** The class-interval is not continuous so frist make the class-interval continuous after determine the class-mark and new score of each teams.

To draw the frequency polygon, balls and score of team A and team B

| Number of balls | Class-mark | Class-size | Team A | A | Team B | B |
|---|---|---|---|---|---|---|
| .5-6.5 | 3.5 | 6 | 2 | $\frac{2}{6} \times 5 = 1.67$ | 5 | $\frac{5}{6} \times 5 = 4.17$ |
| 6.5-12.5 | 9.5 | 6 | 1 | $\frac{1}{6} \times 5 = .83$ | 6 | $\frac{6}{6} \times 5 = 5$ |
| 12.5-18.5 | 15.5 | 6 | 8 | $\frac{8}{6} \times 5 = 6.67$ | 2 | $\frac{2}{6} \times 5 = 1.67$ |
| 18.5-24.5 | 21.5 | 6 | 9 | $\frac{9}{6} \times 5 = 7.5$ | 10 | $\frac{10}{6} \times 5 = 8.33$ |
| 24.5-30.5 | 27.5 | 6 | 4 | $\frac{4}{6} \times 5 = 3.33$ | 5 | $\frac{5}{6} \times 5 = 4.17$ |
| 30.5-36.5 | 33.5 | 6 | 5 | $\frac{5}{6} \times 5 = 4.17$ | 6 | $\frac{6}{6} \times 5 = 5$ |
| 36.5-42.5 | 39.5 | 6 | 6 | $\frac{6}{6} \times 5 = 5$ | 3 | $\frac{3}{6} \times 5 = 2.5$ |
| 42.5-48.5 | 45.5 | 6 | 10 | $\frac{10}{6} \times 5 = 8.33$ | 4 | $\frac{4}{6} \times 5 = 3.33$ |
| 48.5-54.5 | 51.5 | 6 | 6 | $\frac{6}{6} \times 5 = 5$ | 8 | $\frac{8}{6} \times 5 = 6.67$ |
| 54.5-60.5 | 57.5 | 6 | 2 | $\frac{2}{6} \times 5 = 1.67$ | 10 | $\frac{10}{6} \times 5 = 8.33$ |

Table 17

represented along the horizontal and vertical axis respectively. The frequency polygon of given information is given in Fig. (12.2.7).

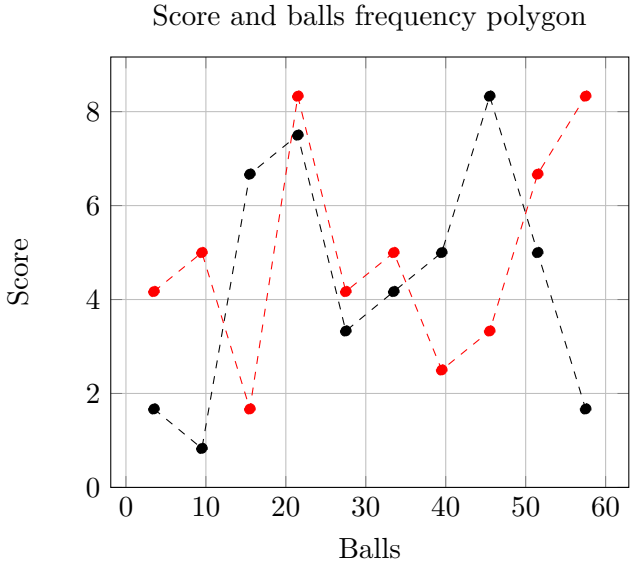



Figure 12.2.7

**Problem 12.2.8.** A random survey of the number of children of various age

groups playing in a park was found as follows table- 18:

| Age (in years) | Number of children |
|---|---|
| 1-2 | 5 |
| 2-3 | 3 |
| 3-5 | 6 |
| 5-7 | 12 |
| 7-10 | 9 |
| 10-15 | 10 |
| 15-17 | 4 |

Table 18

Draw the histogram to represent the data above.

**Solutin:** The class-interval sieze of given data is not equal. So, draw the the histogram we need to find out apropriate proportion length of rectangle. Now we construct new data table for histogram. The minimum class-size of data is 1.

| Age (in years) | Number of children | Width | Redistribution number of children |
|---|---|---|---|
| 1-2 | 5 | 1 | $\frac{5}{1} \times 1 = 5$ |
| 2-3 | 3 | 1 | $\frac{3}{1} \times 1 = 3$ |
| 3-5 | 6 | 2 | $\frac{6}{2} \times 1 = 3$ |
| 5-7 | 12 | 2 | $\frac{12}{2} \times 1 = 6$ |
| 7-10 | 9 | 3 | $\frac{9}{3} \times 1 = 3$ |
| 10-15 | 10 | 5 | $\frac{10}{5} \times 1 = 2$ |
| 15-17 | 4 | 2 | $\frac{4}{2} \times 1 = 2$ |

Table 19

The histogram of given information is represented by Fig.(12.2.8).

**Problem 12.2.9.** 100 surname were randomly picked up from a local telephone direcotory and a frequency distribution of the number of letters in the

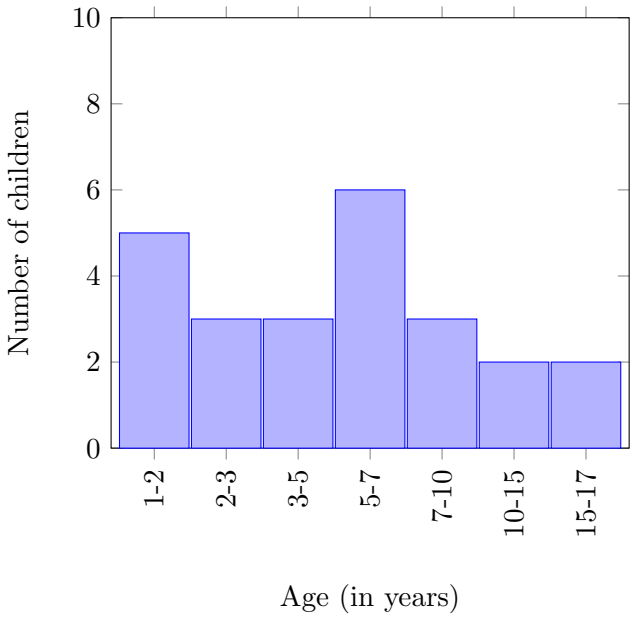



Figure 12.2.8

English alphabet in the surname was found as follows by table-20.

| Number of latter | Number of surname |
|---|---|
| 1-4 | 6 |
| 4-6 | 30 |
| 6-8 | 44 |
| 8-12 | 16 |
| 12-20 | 4 |

Table 20

i Draw a histogram to depict the given information.

ii write the class-interval in which the maximum number of surname lie.

**Solution:** The class-interval of given data is not equal size. So, to draw

histogram we need redistribute of number of surname for equal width rectangle. Construct new table taking minimum class-size 2, as

| Number of latter | Number of surname | Redistribution number of surname |
|---|---|---|
| 1-4 | 6 | $\frac{6}{3} \times 2 = 4$ |
| 4-6 | 30 | $\frac{30}{2} \times 2 = 30$ |
| 6-8 | 44 | $\frac{44}{2} \times 2 = 44$ |
| 8-12 | 16 | $\frac{16}{4} \times 2 = 8$ |
| 12-20 | 4 | $\frac{4}{8} \times 2 = 1$ |
| Total | 100 | |

Table 21

Taking the number of latter and number of surname along horizontal and vertical axis respectively. Then histogram of given information being shown in Fig. (12.2.9).

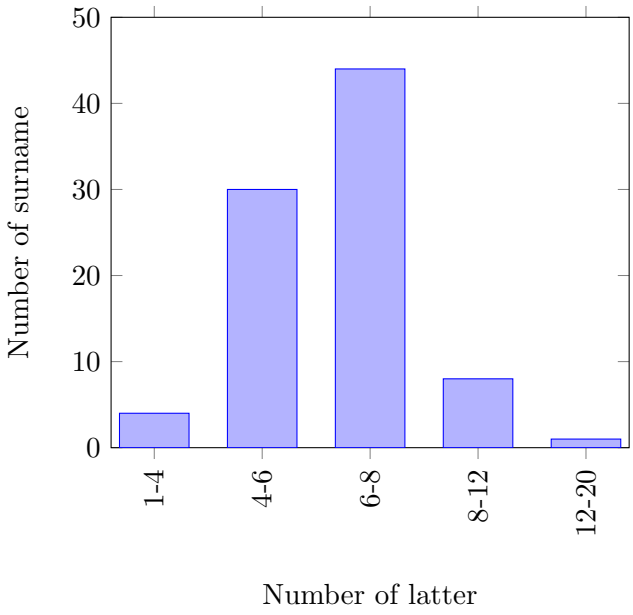



Figure 12.2.9

i The histogram to depict the given information is given by Fig.(12.2.9).

ii From Fig.(12.2.9), the longest rectangle is for class-interval $6 - 8$. So, the maximum number of surname lie in it.

END